Robust Replication of Volatility and Hybrid Derivatives on Jump Diffusions

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Pricing of exotic derivatives: parametric approach

The **parametric approach** to pricing exotic derivatives involves:

- writing a **parametric model** for an underlying $S$ (e.g., Heston, SABR, Hull-White, exponential Lévy),
- **calibrating** the model to liquid calls $C(K)$ and puts $P(K)$,
- using the model and the obtained parameters to price exotics (either analytically or via Monte Carlo).

This approach has a number of **shortcomings**:

- parametric models typically **limited to those that produce closed-form call/put prices**,
- parametric models **cannot fit market data**,
- the models must be **re-calibrated frequently**; calibration is time-consuming.
The **non-parametric approach** to pricing exotics involves:

- writing a **non-parametric model** for an underlying \( S \) (e.g., \( S \) is a positive semimartingale or a continuous positive semimartingale),
- deriving **upper and lower bounds** for exotic derivative prices relative to calls \( C(K) \) and puts \( P(K) \).

This approach has some possible **shortcomings**:

- upper and lower bounds may be **too far apart** for use in practice,
- pricing and hedging strategies **do not allow for dynamic trading of calls and puts**; doing so could narrow no arbitrage bounds.
Pricing of exotic derivatives: semi-parametric approach

The **semiparametric approach** we follow can be outlined as follows:

- write a **semi-parametric model** for an underlying $S$ with **minimal structure**, 
- use the structure to give **unique prices and hedges** for exotics **relative** to liquid calls $C(K)$ and puts $P(K)$.

This approach has a number of **advantages**:

- **compared to parametric models**, semi-parametric models are **more flexible** and more likely to fit market data,
- frequent (re-)calibration **not** needed,
- **compared to non-parametric models** **unique prices** (rather than price bounds) are obtained.
Basic assumptions and notation

Throughout this talk, we make the following assumptions:

- no arbitrage,
- no transactions costs,
- zero interest rates.

We fix a maturity date $T$.

Denote by $S = (S_t)_{0 \leq t \leq T}$ the price of a strictly positive risky asset.

Denote by $X = (X_t)_{0 \leq t \leq T}$ the log price: $X_t = \log S_t$.

Under the above assumptions, put and call prices are given by

$$P(K) = \mathbb{E}(K - S_T)^+, \quad C(K) = \mathbb{E}(S_T - K)^+.$$ 

Here, $\mathbb{E}$ denotes expectation with respect to the market’s chosen pricing measure $\mathbb{P}$.

We assume a call and/or put trades at every strike $K \in (0, \infty)$. 
Non-parametric pricing of European options

Carr and Madan (1998) show that, if \( f \) can be expressed as the difference of convex functions, then for any \( \kappa \in \mathbb{R}^+ \) we have

\[
f(s) = f(\kappa) + f'(\kappa) \left( (s - \kappa)^+ - (\kappa - s)^+ \right) \\
+ \int_0^\kappa dK f''(K)(K - s)^+ + \int_\kappa^\infty dK f''(K)(s - K)^+.
\]

Replacing \( s \) with \( S_T \), setting \( \kappa = S_0 \), and taking an expectation

\[
\mathbb{E} f(S_T) = f(S_0) + \int_0^{S_0} dK f''(K)P(K) + \int_{S_0}^\infty dK f''(K)C(K).
\]

**Takeaway:** the price of any European claim \( \mathbb{E} f(S_T) \) can be expressed relative to puts and calls on \( S_T \).

This result is completely model-free; it makes no assumptions on the \( S \) process.

To price exotics, we need to impose some structure on \( S \) dynamics.
Semi-parametric model

On a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) the asset \(S\) satisfies

\[
dS_t = \sigma_t S_t dW_t + \int_{\mathbb{R}} (e^z - 1) S_t - \tilde{N}(dt, dz),
\]

\[
\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz) dt,
\]

- \(W\) is a **Brownian motion** under \(\mathbb{P}\) with respect to the filtration \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\).
- \(\tilde{N}\) is a **compensated Poisson random measure** with respect to the pricing measure.

The model is **semi-parametric** in that:

- No parametric model for **volatility process** \(\sigma\) (may be non-Markovian, may experience jumps).
- The volatility process \(\sigma\) evolves **independently** of \(W\) and \(\tilde{N}\).
- We must specify **Lévy measure** \(\nu\) parametrically.
Framework allows for asymmetric implied volatility smiles

Imp. vol as a function of log-moneyness-to-maturity for $T = \{1, 2, 3\}$ months.

$$dX_t = \gamma(Z_t)dt + \sqrt{Z_t}dW_t + \int_{\mathbb{R}} z\tilde{N}(dt, dz),$$

$$dZ_t = \kappa(\theta - Z_t)dt + \delta\sqrt{Z_t}dB_t,$$

$$\nu(dz) = \frac{1}{\sqrt{2\pi}s^2} \exp\left(\frac{-(z - m)^2}{2s^2}\right)dz.$$
Types of claims we consider

By Itô's Lemma, the process $X := \log S$ satisfies

\[ dX_t = -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t \]

\[ - \int_{\mathbb{R}} (e^z - 1 - z) \nu(dz) dt + \int_{\mathbb{R}} z \tilde{N}(dt, dz). \]

We wish to price and hedge claims of the form

Payoff at time $T = \varphi(X_T, [X]_T)$,

$[X]_T = \text{realized quadratic variation of } X \text{ up to time } T$.

Examples

Variance Swap : $\varphi(X_T, [X]_T) = [X]_T$,

Volatility Swap : $\varphi(X_T, [X]_T) = \sqrt{[X]_T}$,

Sharpe Ratio : $\varphi(X_T, [X]_T) = (X_T - X_0)/[X]_T$.

We also consider options on Leveraged ETFs, which are path-dependent claims on $X$, but whose payoff cannot be written simply as $\varphi(X_T, [X]_T)$. 
Pricing exponential claims

We will use exponential claims to construct more general claims

exponential claim payoff: \( e^{i\omega X_T + is[X]_T} \)

To this end, the following proposition will be useful.

Proposition

Define \( u : \mathbb{C}^2 \to \mathbb{C} \) and \( \psi : \mathbb{C}^2 \to \mathbb{C} \) as

\[
\begin{align*}
 u(\omega, s) &:= i \left( -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \omega^2 - i\omega + 2is} \right), \\
 \psi(\omega, s) &:= \int_{\mathbb{R}} \nu(dz) \left( e^{i\omega z + isz^2} - 1 - i\omega(e^z - 1) \right).
\end{align*}
\]

Then the joint characteristic function of \((X_T, [X]_T)\) given \( \mathcal{F}_t \) is

\[
\mathbb{E}_t e^{i\omega X_T + is[X]_T} = \frac{e^{(T-t)\psi(\omega, s) + i(\omega - u(\omega, s))X_t + is[X]_t}}{e^{(T-t)\psi(u(\omega, s), 0)}} \mathbb{E}_t e^{iu(\omega, s)X_T}.
\]
Key ingredients of proof

\( X \) can be separated into a continuous component and an independent jump component

\[
\begin{align*}
\ & \ 
\text{d}X_t \ &= \ \text{d}X_t^c + \text{d}X_t^j, \\
\text{d}X_t^c \ &= \ -\frac{1}{2}\sigma_t^2 \text{d}t + \sigma_t W_t, \\
\text{d}X_t^j \ &= \ -\int_{\mathbb{R}} (e^z - 1 - z) \nu(\text{d}z) \text{d}t + \int_{\mathbb{R}} z \tilde{N}(\text{d}t, \text{d}z).
\end{align*}
\]

Carr and Lee (2008) show that the continuous component \((X^c, [X^c])\) satisfies

\[
\mathbb{E}_t e^{i\omega (X^c_T - X^c_t) + is([X^c]_T - [X^c]_t)} = \mathbb{E}_t e^{iu(\omega, s)(X^c_T - X^c_t)}.
\]

The jump component \((X^j, [X^j])\) is a two-dimensional Lévy process with joint characteristic exponent \(\psi\)

\[
\mathbb{E}_t e^{i\omega (X^j_T - X^j_t) + is([X^j]_T - [X^j]_t)} = e^{(T-t)\psi(\omega, s)},
\]

The main result follows from the above and algebra.
Proof

Using results from previous page, we have

$$E_t e^{i \omega (X_T - X_t) + is([X]_T - [X]_t)}$$

$$= E_t e^{i \omega (X^c_T - X^c_t) + is([X^c]_T - [X^c]_t)}$$

$$= E_t e^{i \omega (X^j_T - X^j_t) + is([X^j]_T - [X^j]_t)}$$

$$= E_t e^{i u(\omega, s)(X^c_T - X^c_t)}$$

$$= E_t e^{i u(\omega, s)(X^j_T - X^j_t)}$$

$$= E_t e^{i u(\omega, s)(X_T - X_t)}$$

$$= E_t e^{i u(\omega, s)(X_T - X_t) - (T-t)\psi(\omega, s)}$$

$$= E_t e^{i u(\omega, s)(X_T - X_t) - (T-t)\psi(u(\omega, s), 0)}$$

Thus, we obtain

$$E_t e^{i \omega X_T + is[X]_T} = e^{-i u(\omega, s) X_t} e^{i \omega X_t + is[X]_t} e^{(T-t)\psi(\omega, s)} E_t e^{i u(\omega, s) X_T}.$$
Pricing power-exponential claims

We can use previous result to price power-exponential claims

\[
E_t X^n_T [X]^m_T e^{i\omega X_T + is[X]}_T
\]

\[
= (-i\partial_\omega)^n(-i\partial_s)^m E_t e^{i\omega X_T + is[X]}_T
\]

\[
= (-i\partial_\omega)^n(-i\partial_s)^m \frac{e^{(T-t)\psi(\omega,s)+i(\omega-u(\omega,s))X_t+is[X]}_t}{e^{(T-t)\psi(u(\omega,s),0)}} E_t e^{iu(\omega,s)X_T}
\]

\[
=: F(\omega,s,X_t,[X]_t)
\]

\[
= \sum_{j=0}^{n} \sum_{k=0}^{m} \binom{n}{j} \binom{m}{k} (-i\partial_\omega)^j(-i\partial_s)^k F(\omega,s,X_t,[X]_t)
\]

\[
\mathcal{F}_t\text{-measurable}
\]

\[
\times E_t (-i\partial_\omega)^{n-j}(-i\partial_s)^{m-k} e^{iu(\omega,s)X_T}
\]

Eur. claim
Example: variance swap

We plot $g(\log s)$ as a function of $s$ where

$$
Eg(\log S_T) = E[\log S]_T, \quad \nu(dz) = \lambda\delta_m(z)dz, \quad T = 0.25.
$$

Left : $\lambda = 1.00, \quad m = \{-2.00, 0, 2.00\}$,
Right : $m = -2.00, \quad \lambda = \{1.00, 2.00, 3.00\}$,
Pricing fractional powers of $[X]_T$

Using the following integral representation

$$v^r = \frac{r}{\Gamma(1 - r)} \int_0^\infty dz \frac{1}{z^{r+1}} \left(1 - e^{-zv}\right), \quad 0 < r < 1,$$

we have (taking $X_0 = 0$ for simplicity)

$$\frac{\Gamma(1 - r)}{r} E[X]^r_T$$

$$= \int_0^\infty dz \frac{1}{z^{r+1}} \left( E1 - Ee^{-z[X]_T} \right)$$

$$= E \int_0^\infty dz \frac{1}{z^{r+1}} \left( e^{iu(0,0)X_T} - \frac{e^{T\psi(0,iz)}}{e^{T\psi(u(0,iz),0)}} e^{iu(0,iz)X_T} \right)$$

$$= \frac{\Gamma(1 - r)}{r} Eg(X_T),$$
Example: volatility swap

Effect of jump size

Effect of jump intensity

We plot \( g(\log s) \) as a function of \( s \) where

\[
\mathbb{E}g(\log S_T) = \mathbb{E}\sqrt{[\log S]_T}, \quad \nu(dz) = \lambda \delta_m(z)dz, \quad T = 0.25.
\]

Left: \[ \lambda = 1.00, \quad m = \{-1.25, 0.00, 1.25\}, \]
Right: \[ m = -1.25, \quad \lambda = \{1.00, 2.00, 3.00\}, \]
Pricing ratio claims $X_T / ([X]_T + \varepsilon)^r$

Using the integral representation

$$\frac{1}{(v + \varepsilon)^r} = \frac{1}{r \Gamma(r)} \int_0^\infty dz \, e^{-z^{1/r}(v + \varepsilon)}, \quad r > 0,$$

we have

$$\mathbb{E} \frac{X_T e^{ipX_T}}{([X]_T + \varepsilon)^r} = \frac{1}{r \Gamma(r)} \int_0^\infty dz \, \mathbb{E} X_T e^{ipX_T - z^{1/r}([X]_T + \varepsilon)}$$

$$= \frac{1}{r \Gamma(r)} \int_0^\infty dz \, e^{-z^{1/r}\varepsilon} (-i \partial_p) \mathbb{E} e^{ipX_T - z^{1/r}([X]_T + \varepsilon)}$$

exponential claim

$$= \frac{1}{r \Gamma(r)} \mathbb{E} \int_0^\infty dz \, e^{-z^{1/r}\varepsilon} (-i \partial_p) \frac{e^{T\psi(p, iz^{1/r})}}{e^{T\psi(u(p, iz^{1/r}), 0)}} e^{iu(p, iz^{1/r})X_T}$$

$$=: \mathbb{E} g(X_T).$$
Example: realized Sharpe ratio

\[ \lambda = 1.0, \ m = -0.675 \]

\[ \lambda = 2.0, \ m = -0.675 \]

\[ \lambda = 1.0, \ m = 0.675 \]

\[ \lambda = 2.0, \ m = 0.675 \]

We plot \( g(\log s) \) as a function of \( s \) where \( \varepsilon = 0.001 \) and

\[ \mathbb{E}g(\log S_T) = \mathbb{E}X_T/\sqrt{[\log S]_T} + \varepsilon, \quad \nu(dz) = \lambda \delta_m(z)dz. \]
Leveraged ETFs

The relationship between an Leveraged Exchange Traded Fund (LETF) $L = e^Y$ and the underlying Exchange Traded Fund ETF $S = e^X$ is

$$\frac{dL_t}{L_{t-}} = \beta \frac{dS_t}{S_{t-}},$$

where $\beta \in \{-2, -1, 2, 3\}$ is the leverage ratio.

The value of $Y_T$ depends on the path of $X$ as follows

$$dY_t = dY^c_t + dY^j_t,$$

$$dY^c_t = \beta dX^c_t + \frac{1}{2} \beta (1 - \beta) d[X^c]_t,$$

$$dY^j_t = -\int_{\mathbb{R}} \left( \beta (e^z - 1) - \log \left( \beta (e^z - 1) + 1 \right) \right) \nu(dz) dt$$

$$+ \int_{\mathbb{R}} \log \left( \beta (e^z - 1) + 1 \right) \tilde{N}(dt, dz).$$
Characteristic Function of $Y_T$

Despite dependence on path of $X$ we can relate the characteristic function of $Y_T$ to the characteristic function of $X_T$ only:

**Proposition**

Define $\chi : \mathbb{C} \to \mathbb{C}$ by

$$\chi(q) := \int_{\mathbb{R}} \nu(dz) \left( (\beta(e^z - 1) + 1)^{iq} - 1 - iq\beta(e^z - 1) \right).$$

Then the characteristic function of $(Y_T - Y_t)$, conditional on $\mathcal{F}_t$, is

$$\mathbb{E}_t e^{iq(Y_T - Y_t)} = \frac{e^{(T-t)\chi(q)}}{e^{(T-t)\psi(u(q\beta, q\frac{1}{2}\beta(1-\beta)), 0)}},$$

where $u$ and $\psi$ as defined previously.

The path-dep. claim = $e^{(T-t)\chi(q)}$

Eur. claim = $\mathbb{E}_t e^{iu(q\beta, q\frac{1}{2}\beta(1-\beta))(X_T - X_t)}$,

$\mathcal{F}_t$-measurable
Proof

Using

\[ \mathbb{E}_t e^{i q (Y^j_{T} - Y^j_{t})} = e^{(T-t) \chi(q)}, \]  
\[ \mathbb{E}_t e^{i \omega (X^j_{T} - X^j_{t}) + i s ([X^j]_T - [X^j]_t)} = e^{(T-t) \psi(\omega,s)}, \]  

and independence of continuous and jump components, we have

\[ \mathbb{E}_t e^{i q (Y_{T} - Y_{t})} = \mathbb{E}_t e^{i q (Y^c_{T} - Y^c_{t})} \mathbb{E}_t e^{i q (Y^j_{T} - Y^j_{t})} \]  
\[ = \mathbb{E}_t e^{i q \beta (X^c_{T} - X^c_{t}) + i q \frac{1}{2} \beta (1-\beta) ([X^c]_T - [X^c]_t)} e^{(T-t) \chi(q)} \]  
\[ = \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^c_{T} - X^c_{t})} e^{(T-t) \chi(q)} \]  
\[ = \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^c_{T} - X^c_{t})} \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^j_{T} - X^j_{t})} e^{(T-t) \chi(q)} \]  
\[ = \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^c_{T} - X^c_{t})} \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^j_{T} - X^j_{t})} e^{(T-t) \chi(q)} \]  
\[ = \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^c_{T} - X^c_{t})} e^{(T-t) \chi(q)} \]  
\[ = \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^c_{T} - X^c_{t})} e^{(T-t) \psi(u(q \beta, q \frac{1}{2} \beta (1-\beta)),0)} \]  
\[ = \mathbb{E}_t e^{i u (q \beta, q \frac{1}{2} \beta (1-\beta)) (X^c_{T} - X^c_{t})} e^{(T-t) \psi(u(q \beta, q \frac{1}{2} \beta (1-\beta)),0)}. \]  

(by (1))

(by (2))
Pricing general claims on $Y_T$

Let $\hat{\varphi}$ be the (possibly generalized) Fourier transform of $\varphi$

$$\hat{\varphi}(q) = \frac{1}{2\pi} \int_{\mathbb{R}} dy \, e^{-iqy} \varphi(y).$$

The price of a claim with payoff $\varphi(Y_T)$ can be obtained as follows

$$E_t \varphi(Y_T)$$

$$= \int_{\mathbb{R}} dq \, \hat{\varphi}(q) e^{iqY_t} E_t e^{iq(Y_T - Y_t)}$$

$$= \int_{\mathbb{R}} dq \, \hat{\varphi}(q) e^{iqY_t} \frac{e^{(T-t)\chi(q)}}{e^{(T-t)\psi(u(q\beta,q\frac{1}{2}\beta(1-\beta)),0)}} E_t e^{iu(q\beta,q\frac{1}{2}\beta(1-\beta))(X_T - X_t)}$$

$$=: E_t g(X_T; X_t, Y_t),$$

Eur. claim
Example: Calls on $L_T$

We plot $g(\log s; x, y)$ as a function of $s$ where

$$
\mathbb{E}g(\log S_{T}; X_0, Y_0) = \mathbb{E}(L_T - K)^+, \quad \nu(dz) = \lambda \delta_m(z)dz.
$$

where $K = 1.0$, $T = 1/4$, $X_0 = Y_0 = 0.0$, $m = -0.4$ and $\lambda = 2.0$.

Left : $\beta = \{1.0, 2.0, 3.0\}$,  
Right : $\beta = \{-1.0, -2.0, -3.0\}$,
Hedging Exponential claims

The value of an exponential claim at any time \( t \leq T \) is

\[
\mathbb{E}_t e^{i \omega X_T + is[X]_T} = A_t Q_t^{(u)}, \quad u \equiv u(\omega, s),
\]

where we have defined

\[
A_t := e^{i(\omega-u)X_t + is[X]_t} \frac{e^{(T-t)\psi(\omega,s)}}{e^{(T-t)\psi(u,0)}},
\]

\[
Q_t^{(u)} := \mathbb{E}_t e^{iuX_T}.
\]

Strategy for deriving hedging strategy is to take the differential

\[
d(A_t Q_t^{(u)}) = A_t dQ_t^{(u)} + Q_t^{(u)} dA_t + d[A, Q^{(u)}]_t,
\]

and show that the right-hand side can be expressed as a self-financing portfolio of traded assets’’

- the stock \( S \)
- zero-coupon bonds \( B \)
- European exponential claims \( Q^{(q)} \) where \( q \in \mathbb{C} \).
Key ingredients in derivation

- Jump in value of European claim $Q_{t}(q) = E_t e^{i q X_T}$ is

$$\Delta Q_{t}(q) = Q_{t-}(e^{i q \Delta X_t} - 1) + \text{jump due to } \Delta \sigma_t.$$ 

- Then we have the following symmetry

$$R_{t}^{(q)} Q_{t}^{(q)} = R_{t}^{(-i-q)} Q_{t}^{(-i-q)},$$

where the process $R^{(q)}$ is given by

$$R_{t}^{(q)} = e^{-i q X_t + (T-t)\psi(-i-q,0)}.$$
Explicit strategy

Define $\Gamma^{(u)}$ and $\Omega^{(q)}$ where $q \in \mathbb{C}$ by

$$d\Gamma^{(u)}_t := A_{t^-}Q^{(u)}_{t^-} \int_{\mathbb{R}} \left( e^{i\omega z + is^2} - e^{iuz} - i(\omega - u)(e^z - 1) \right) N(dt, dz),$$

$$d\Omega^{(q)}_t := R^{(q)}_{t^-}Q^{(q)}_{t^-} \int_{\mathbb{R}} \left( -e^{iqz} + 1 + iq(e^z - 1) \right) N(dt, dz).$$

Let $q \in \mathbb{C}^m$. Suppose there exists predictable $H \in \mathbb{C}^m$ satisfying

$$0 = \Delta \Gamma^{(u)}_t + \sum_{j=1}^{m} H^{(j)}_t \left( \Delta \Omega^{(q_j)}_t - \Delta \Omega^{(-i-q_j)}_t \right).$$

Then we have (traded assets in blue)

$$d(A_t Q^{(u)}_t) = A_t dQ^{(u)}_t + i(\omega - u)\frac{A_{t^-}Q^{(u)}_{t^-}}{S_{t^-}}dS_t$$

$$+ \sum_{j=1}^{m} H^{(j)}_t \left( R^{(q_j)}_{t^-} dQ^{(q_j)}_t - R^{(-i-q_j)}_{t^-} dQ^{(-i-q_j)}_t \right)$$

$$+ \sum_{j=1}^{m} H^{(j)}_t (1 - 2iq_j)\frac{R^{(q_j)}_{t^-} Q^{(q_j)}_{t^-}}{S_{t^-}}dS_t.$$
Conclusion

- We have presented a **semiparametric model** for an asset $S$
  - The volatility process $\sigma$ is **non-parametric**; may be non-Markovian (e.g., driven by fBM) and may experience jumps
  - The jumps of $X$ must be specified **parametrically** via Lévy measure $\nu$
  - Model allows for **asymmetric implied volatility** smiles
- We have shown how to price path-dependent claims relative to European calls and puts
  - variance-style claims
  - hybrid claims on price and volatility
  - options on LETFs
- We have shown how to **replicate** exponential claims with a self-financing portfolio of traded assets.