Fundamentally, financial mathematics is concerned with modeling the random movements of underlyers (e.g., stocks, indexes, interest rates, etc.) and developing the computational tools that are needed to price and hedge the risks associated with these movements. One of the principal modeling challenges is that different financial underlyers exhibit an array of different behaviors (e.g., stochastic volatility, mean-reversion, jumps, etc.). Moreover, for any given underlyer, there is no consensus on how to properly capture its dynamics either under the physical measure or the risk-neutral pricing measure. With this in mind, my research focuses on developing computational methods that are applicable to large classes of models. Speaking broadly, my work can be divided into three categories (i) coefficient polynomial expansions, (ii) multiscale expansions, and (iii) semi-parametric methods. Below, I highlight some of my key contributions in these three areas.

1 Coefficient Polynomial Expansions

Perhaps the most significant theme of my research over the past three to four years has been on developing analytical approximate solutions to various partial differential equations (PDEs) that arise in mathematical finance. The main approach, developed jointly with my co-authors Andrea Pascucci and Stafano Pagliarani, has been to make use of *Coefficient Polynomial Expansions* (CPEs). The CPE method has led to a series of papers on topics such as contingent claim valuation, analysis of implied volatility, utility maximization and indifference pricing. Currently, Andrea, Stefano and I are writing a book on the CPE method.

1.1 Parabolic PDEs

The CPE method is most easily understood in the diffusion setting. To begin, let us fix a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. Consider a d-dimensional Markov diffusion $X = (X_t)_{t\geq 0}$, which satisfies the following stochastic differential equation (SDE)

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \tag{1}$$

where W is a standard m-dimensional $(\mathbb{F}, \mathbb{P})$-Brownian motion, the drift vector μ maps $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and the volatility matrix σ maps $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$. Define a function u by the following conditional expectation

$$u(t, x) := \mathbb{E}_{t,x}\Big(\mathrm{e}^{-\int_t^T \lambda(s,X_s)\mathrm{d}s}\varphi(X_T)\Big), \tag{2}$$

where λ maps $\mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$. The function u arises frequently in mathematical finance, where it can be thought of as the price of a European claim that pays $\varphi(X_T)$ at time T. If $u \in C^{1,2}(\mathbb{R}_+, \mathbb{R}^d)$ and satisfies certain growth conditions, then u satisfies the Kolmogorov Backward Equation (KBE), a linear parabolic PDE of the form

$$0 = (\partial_t + \mathcal{A})u, \qquad u(T, \cdot) = \varphi, \qquad \mathcal{A} = \sum_{|\alpha|\leq 2} a_\alpha(t, x)\partial_x^\alpha, \tag{3}$$

where, for compactness, we have introduced standard multi-index notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$. As an explicit solution of the KBE (3) is generally not available, if one is interested in the solution u, one must resort to approximation methods.

In [LPP15a], we propose to find an approximation solution of (3) by expanding the coefficients of $\mathcal{A}$ in polynomial basis functions. Formally, we have

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n, \qquad \mathcal{A}_n := \sum_{|\alpha|\leq 2} a_{\alpha,n}(t, x)\partial_x^\alpha,$$

where the coefficients $(a_{\alpha,n})$ depend polynomially on x and the zeroth order coefficients $(a_{\alpha,0})$ are constant in x (i.e., they depend only on t). We provide here two examples polynomial expansions.

Example 1 (Taylor polynomial expansion)**.** Assume the coefficients $a_\alpha(t, \cdot)$ are analytic for all t. Then, for any fixed $\bar{x} \in \mathbb{R}^d$, define

$$a_{\alpha,n}(t, x) = \sum_{|\beta|=n} \frac{\partial^\beta a_\alpha(t, \bar{x})}{\beta!}(x - \bar{x})^\beta,$$

Note that $a_{\alpha,n}(t,\cdot)$ is the nth order term of the Taylor expansion of $a_\alpha(t,\cdot)$ about the point $\bar{x}$.

Example 2 (Orthogonal polynomial expansion)**.** Suppose $a_\alpha(t,\cdot) \in L^2(\mathbb{R}^d, \mathfrak{m})$ for all t, where $\mathfrak{m}$ is a measure on $\mathbb{R}^d$. Let $(\mathbf{P}_\beta)$ be a complete set of orthogonal polynomials in $L^2(\mathbb{R}^d,\mathfrak{m})$, properly normalized so that $\langle \mathbf{P}_\alpha, \mathbf{P}_\beta \rangle_\mathfrak{m} = \delta_{\alpha,\beta}$. Here, $\langle \cdot,\cdot \rangle_\mathfrak{m}$ is the usual L^2 inner product on $\mathbb{R}^d$ weighted by $\mathfrak{m}$ and $\delta_{\alpha,\beta}$ is the Kronecker delta. Define

$$a_{\alpha,n}(t,x) = \sum_{|\beta|=n} \langle \mathbf{P}_\beta, a_\alpha(t,\cdot) \rangle_\mathfrak{m} \mathbf{P}_\beta(x).$$

Note that $a_{\alpha,n}(t,\cdot)$ is the projection of $a_\alpha(t,\cdot)$ of the subspace of $L^2(\mathbb{R}^d,\mathfrak{m})$ spanned by $(\mathbf{P}_\beta)_{\beta=n}$.

Any polynomial expansion of $\mathcal{A}$ can be used to find an approximate solution u of the KBE (3) as follows. Suppose, formally, that the solution to (3) can be expanded as an infinite sum: $u = \sum_n u_n$. Then, inserting this expansion for u and a polynomial expansion of $\mathcal{A}$ into (3) and collecting terms whose subscripts sum to like order, we obtain

$$\begin{aligned} 0 &= (\partial_t + \mathcal{A}_0)u_0, && u_0(T,\cdot) = \varphi, \\ 0 &= (\partial_t + \mathcal{A}_0)u_n + \sum_{k=1}^n \mathcal{A}_k u_{n-k}, && u_n(T,\cdot) = 0, && n \geq 1. \end{aligned}$$

After solving the above sequence of nested PDEs, the nth order approximation of u is then given by $\bar{u}_n := \sum_{i=0}^n u_n$. The CPE method enjoys a number of properties, which we state here.

Explicit families of solutions: We provide an explicit solution for every term u_n in the sequence $(u_n)_{n\geq 0}$. In fact, we find *families* of solutions – one for each polynomial expansion of $\mathcal{A}$.

Rigorous asymptotic accuracy: For the Taylor expansion of $\mathcal{A}$ described in Example 1, we prove that the nth order approximation of u satisfies the following asymptotic accuracy property

$$\sup_{x\in\mathbb{R}^d} |u(t,x) - \bar{u}_n(t,x)| = \mathcal{O}\left((T-t)^{(n+1+p)/2} \right), \qquad \text{as } t \nearrow T, \tag{4}$$

where $p \in [0,2]$ is a constant that depends on the smoothness of the terminal data φ.

Numerical efficiency: For any polynomial expansion of $\mathcal{A}$, the nth order approximation $\bar{u}_n$ can be computed as a convolution with a Gaussian kernel. For an options-pricing standpoint, the approximation price $\bar{u}_n$ can be computed with the same numerical efficiency as the corresponding Black-Scholes price.

Generality: The nth order approximation formula $\bar{u}_n$ and the accuracy result (4) hold for any diffusion whose coefficients are $C^{n+1}(\mathbb{R}^d)$ and satisfy a *local* uniform ellipticity condition (global ellipticity is *not* required). The results therefore can be applied to well-known equity models whose generators are not uniformly elliptic such as, e.g., Heston, the Constant Elasticity of Variance (CEV), and Stochastic Alpha Beta Rho (SABR).

Local and global properties: By expanding the coefficients of $\mathcal{A}$ in an arbitrary polynomial basis functions we are able to examine both local properties of u (e.g., using the Taylor series basis described in Example 1) and global properties of u (e.g., using orthogonal polynomial basis functions as described in Example 2).

The CPE method has been applied to a variety of problems arising in mathematical finance. For example, in [LPP15b, LPP15d, LPP15e], we find approximate solutions to partial integro-differential equations when one wishes to compute u in (2) and X is a Lévy-type process of the form

$$\mathrm{d}X_t = \mu(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t + \int_{\mathbb{R}^d} z\widetilde{N}(t,X_{t-},\mathrm{d}t,\mathrm{d}z),$$

where $\widetilde{N}(t,x,\mathrm{d}t,\mathrm{d}z) = N(t,x,\mathrm{d}t,\mathrm{d}z) - \nu(t,x,\mathrm{d}z)\mathrm{d}t$ is a compensated Lévy measure on $\mathbb{R}^d$ with time and state-dependent Lévy kernel ν. In [LS16], we find the approximate value function (which solves a fully nonlinear HJB equation) and optimal investment strategy for an investor who wishes to maximize expected utility from terminal wealth when a risky asset exhibits both stochastic and local volatility effects. And currently, I am investigating how the CPE method can be applied to find approximate solutions of quasilinear PDEs that arise when one wishes to solve forward-backward differential equations (FBSDEs).

1.2 Implied Volatility for Local-Stochastic Volatility Models

Consider a diffusion X of the form (1). If $u(t,x;T,k) := \mathbb{E}_{t,x}(\mathrm{e}^{X_T^1} - \mathrm{e}^k)^+$ is the price of a European call, then the *implied volatility* corresponding to u is the unique positive solution σ of the equation

$$u(t,x;T,k) = u^{\mathrm{BS}}(t,x;T,k,\sigma),$$

where u^{BS} denotes the Black-Scholes price of a call with strike e^k and maturity T as computed assuming a Black-Scholes volatility of σ. Quoting call and put prices in units of implied volatility has become standard practice in the financial industry because options with different strikes and maturities may have wildly different dollar values but will often have similar implied volatilities. Indeed, banks calibrate equity models to the market's implied volatility surface rather than to prices directly for precisely this reason. Because of the important role implied volatility plays, a great deal of research has focused on developing closed-form approximations for model-induced implied volatilities.

In [LPP15c], we consider a general class of multi-factor local-stochastic volatility models of the form (1). Using the CPE method, we derive an explicit approximation $\bar{u}_n = \sum_{i=1}^n u_i$ for call option prices. We than translate the approximation for call prices into an approximation $\bar{\sigma}_n = \sum_{i=1}^n \sigma_i$ of implied volatility by solving

$$u_0 + u_1 + u_2 + \ldots = u^{\mathrm{BS}}(\sigma_0 + \sigma_1 + \sigma_2 + \ldots),$$

order by order for σ_0, σ_1, σ_2, etc. (above, the arguments (t,x,T,k) are committed for brevity). The key contributions of this paper are as follows:

Explicit implied volatility approximations: The nth order implied volatility expansion $\bar{\sigma}_n$ is an nth order polynomial in log-*moneyness* $(x-k)$ and time-to-maturity $(T-t)$. In particular, the coefficients of the implied volatility expansion are expressed as derivatives of the diffusion coefficients (a_α) at the starting point of the diffusion $X_t = x$. We therefore provide a directly link between the local geometry of the underlying diffusion and the implied volatility surface.

Rigorous Asymptotic Accuracy: The nth order approximation of implied volatility $\bar{\sigma}_n$ satisfies the following asymptotic accuracy result: for any $\lambda > 0$ we have

$$|\sigma(t,x;T,k) - \bar{\sigma}_n(t,x;T,k)| = \mathcal{O}\left((T-t)^{(n+1)/2}\right), \qquad \text{as } t \nearrow T, \qquad |k-x| \leq \lambda\sqrt{T-t}, \tag{5}$$

where we have indicated explicitly the dependence of σ on $(t,x;T,k)$.

Generality: The nth order implied volatility expansion $\bar{\sigma}_n$ and accuracy result (5) are valid for any local-stochastic volatility model whose coefficients are $C^{n+1}(\mathbb{R}^d)$ and satisfy a *local* uniform ellipticity condition. The results therefore can be applied to all of the most local volatility models (e.g. CEV [Cox75], quadratic [And11]), stochastic volatility models (e.g., Heston [Hes93], three-halves [BB12]) and local-stochastic volatility models (e.g., SABR [HKLW02], λ-SABR [HL05]). To the best of our knowledge, there is no other implied volatility expansion that can be applied to such a large class of models.

The results of our work have been recognized not only in academia but also in industry. An article about the expansion has appeared in the practitioner-aimed journal Wilmott [Sta14], and the implied volatility expansion has been implemented in QuantLib, the open-source C++ quantitative finance library.

I have written two other papers in which the CPE method is used to analyze features of implied volatility. In [LLP16], we derive an explicit implied volatility expansion for options written on a Leveraged Exchange Traded Fund L, whose dynamics are related to the underlying Exchange Traded Fund S via $\mathrm{d}L_t/L_t = \beta \mathrm{d}S_t/S_t$, where $\beta \in \mathbb{Z}$ is the leverage ratio. What is notable about this work is that options on L are in fact *path-dependent* options on S. In a separate paper [Lor16], I derive an explicit approximations for the implied volatility surfaces induced by buyer's and seller's indifference prices. The key challenge here was to deal with the nonlinear indifference pricing mechanism.

2 Multiscale Expansions

The use of *multiscale expansions* in mathematical finance has been pioneered by Fouque, Papanicolaou, Sircar and Sølna. Motivated by empirical studies, which have identified a fast time scale in stock price volatility on the order of days as well as a slow scale on the order of months [CRGGT03, Hil05, LeB01], they model a risky asset $S = e^X$ by

$$dX_t = -\tfrac{1}{2} f^2(Y_t, Z_t)dt + f(Y_t, Z_t)dW_t,$$

where W is a Brownian motion, Y is a *fast* process with a characteristic time-scale $\varepsilon \ll 1$ and Z is a *slow* process with a characteristic time-scale $1/\delta \gg 1$. Using a combined singular-regular perturbation analysis, they derive approximations for both European option prices and the corresponding implied volatilities. The multiscale approach has been successfully applied to options on interest rates, exotic options, credit derivatives, utility maximization and American options (see [FPSS11] for an overview). In my earliest work, I develop a unified framework, which can be used to find multiscale approximations for a variety of problems in mathematical finance. I also broaden the scope of multiscale methods by applying them to Lévy-type models; previously multiscale methods were limited to the diffusion setting.

2.1 Spectral Methods

In [Lor14], I consider a class of multiscale diffusions, whose dynamics under a pricing measure $\mathbb{P}$ are of the form

$$\begin{aligned} dX_t &= \mu(X_t, Y_t, Z_t)dt + \sigma(X_t, Y_t, Z_t)dW_t^X, \\ dY_t &= \left(\frac{1}{\varepsilon}\alpha(Y_t) - \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)\Lambda(Y_t, Z_t)\right)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y_t)dW_t^Y, && (6) \\ dZ_t &= \left(\delta c(Z_t) - \sqrt{\delta} g(Z_t)\Gamma(Y_t, Z_t)\right)dt + \sqrt{\delta} g(Z_t)dW_t^Z, && (7) \end{aligned}$$

where (W^X, W^Y, W^Z) are correlated Brownian motions. The process X could represent a variety of things: the price of an asset, the short rate of interest, an economic indicator, etc.. In this setting, the prices of credit derivatives, European and barrier claims, defaulable bonds, interest rate derivatives and a variety of other claims written on X can be expressed as

$$u^{\varepsilon,\delta}(t, x, y, z) = \mathbb{E}_{t,x,y,z}\left(\mathbb{1}_{\{\tau > T\}} e^{\int_t^T \lambda(X_s)ds} \varphi(X_T)\right), \qquad (8)$$

where τ is a stopping time, which could be the first exit time of X from some interval $I \subset \mathbb{R}$, or the first time a process of the form $\int_0^t \gamma(X_s)ds$ exceeds an independent exponentially distributed random variable $\mathcal{E}$. The key contribution of [Lor14] is to show that the approximate price $u_0 + \sqrt{\varepsilon}u_{1,0} + \sqrt{\delta}u_{0,1}$ of *all* claims of the form (8) can be obtained solving a single generalized eigenvalue problem, thereby unifying many of the distinct pricing and accuracy results in the book [FPSS11] into a single framework. My results also provide a means of extending many of the eigenfunction expansion results for scalar diffusions (see, e.g., [DL03] for an overview) to the multiscale setting. Note that this is a non-trivial extension as, unlike the generator of a scalar diffusion, the generator of the multiscale diffusion (X, Y, Z) is *not*, in general, a self-adjoint operator.

2.2 Multiscale Lévy-type Processes

It is known that jumps are required in order to fit the steep smile of implied volatility for short-maturity equity options (see [CT04], Chapter 15). This has motivated researchers to model risky assets $S = e^X$ as exponential Lévy process. Yet, because Lévy processes have stationary and independent increments, exponential Lévy models cannot fit the term-structure of implied volatility. With this in mind, in [LLC15], we model a risky asset $S = e^X$ as an exponential Lévy-type process of the form

$$dX_t = \mu(Y_t, Z_t)dt + \sigma(Y_t, Z_t)dW_t^X + \int_{\mathbb{R}} s\widetilde{N}(Y_{t-}, Z_{t-}, dt, ds),$$

where $\widetilde{N}(y, z, dt, ds) = N(y, z, dt, ds) - \nu(y, z, ds)dt$ is a compensated Poisson random measure with a state-dependent Lévy kernel ν, and Y and Z are fast and slow diffusions, respectively, given by (6)-(7). In addition to finding approximate prices for European options and deriving rigorous asymptotic accuracy results, we show that the model-induced implied volatility surface provides a tight fit to the market's implied volatility surface for options on the S&P500 with maturities ranging from one month to three years. Note that we calibrate to all strikes and maturities simultaneously rather than calibrating maturity-by-maturity. An important feature of our framework is that it enables one to include stochastic jump-size and stochastic jump-intensities into pure-jump exponential Lévy models such as Variance Gamma [MCC98] and CGMY/Kobol [CGMY02, BL00], thereby mimicking stochastic volatility affects.

3 Semiparametric Methods

A classical question in financial mathematics is the following: Let $S = (S_t)_{t\geq 0}$ be the price of stock. Given the values of T-maturity calls and puts at every strike $K \in (0, \infty)$, how can one price and hedge a contingent claim with payoff $F[S]$, where F is a functional of the path of S over the interval $[0, T]$? Together with my co-authors Peter Carr and Roger Lee, we have developed a *semi-parametric* approach to answering this question. A general outline of our semi-parametric approach is as follows:

Step 1: Assume a semiparametric model for S. Some aspects of S are specified parametrically while other aspects of S are left entirely nonparametric. In writing a semiparametric model for S, the aim is to impose the minimal structure necessary to carry out the Steps 2–4 below.

Step 2: Relate the value of a path-dependent claim with payoff $F[S]$ to the value of a path-independent European claim with payoff $f(S_T)$. A typical result will be of the form $\mathbb{E}_t F[S] = \mathbb{E}_t f(S_T)$, where f will depend on the semiparametric model for S and the functional F of interest.

Step 3: Use the following classical result from [CM98]

$$\mathbb{E}_t f(S_T) = f(S_t)B_t + \int_0^{S_t} \mathrm{d}K f''(K)P_t(K) + \int_{S_t}^{\infty} \mathrm{d}K f''(K)C_t(K). \tag{9}$$

to determine the value $\mathbb{E}_t f(S_T)$ of the European claim (and thus the value $\mathbb{E}_t F[S]$ of the path-dependent claim) in terms the bond price B_t, put prices $P_t(K)$ and call prices $C_t(K)$.

Step 4: If possible, find a hedging portfolio $\Pi = (\Pi_t)_{0\leq t\leq T}$ that replicates the path-dependent claim payoff: $\Pi_T = F[S]$. The hedging portfolio will generally involve dynamic trading of bonds B, the underling S, European puts $P(K)$ and European calls $C(K)$.

The semiparametric approach outlined above combines desirable features of parametric and nonparametric approaches. Like parametric approaches, the semiparametric approach yields explicitly computable claim prices and hedging strategies. Similar to nonparametric approaches, the pricing and hedging results of the semiparametric approach are robust to some (but not all) sources of model mis-specification risk.

3.1 Barrier-Style Claims on Price and Volatility

In [CLL15], on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we consider a risky asset $S = \mathrm{e}^X$, which is strictly positive and has continuous sample paths. Assuming zero interest rates for simplicity, in order to preclude arbitrage the asset S must be a martingale under the (market's chosen) pricing measure $\mathbb{P}$. As such, there exists a non-negative, $\mathbb{F}$-adapted stochastic process $\sigma = (\sigma_t)_{0\leq t\leq T}$ such that

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t,$$

where W is a Brownian motion with respect to the pricing measure $\mathbb{P}$ and the filtration $\mathbb{F}$. We do not assume any model for the *volatility process* σ. Rather, we assume only that σ is right-continuous and $\mathbb{F}$-adapted and that it evolves independently of W. Note that σ may experience jumps and is not required to be Markovian. Thus, our results include so-called *rough* volatility models driven by fractional Brownian motion. In this setting, we consider barrier-style claims on price and volatility with three types of payoffs

$$\begin{aligned} &\text{knock-out}: && \mathbb{1}_{\{\tau > T\}}\varphi(X_T, [X]_T), \\ &\text{knock-in}: && \mathbb{1}_{\{\tau \leq T\}}\varphi(X_T - X_{\tau^*}, [X]_T - [X]_{\tau^*}), \\ &\text{rebate}: && \mathbb{1}_{\{\tau \leq T\}}\varphi([X]_{\tau^*}), \end{aligned}$$

where $\varphi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a payoff function and τ^* is the smaller of T and the first exit time τ of some interval $I \subset \mathbb{R}$

$$\tau^* := \tau \wedge T, \qquad \tau = \inf\{t \geq 0 : X_t \notin I\}.$$

Note that by taking $I = (L, U)$ we are able to handle double-barrier claims and by taking $I = (L, \infty)$ or $I = (-\infty, U)$ we are able to handle single-barrier claims. Under certain growth and smoothness restrictions on the payoff function φ we are able to derive unique prices for each of the above claims relative to the T-maturity calls and puts. We are also able to find hedging portfolios that replicate the claim payoffs with probability one.

3.2 Options on Realized Variance for Models with Jumps

There is evidence from both time series [Era04] and options data [CT04, Ch. 15] that stock prices exhibit jumps. With this in mind, in [CLL17a], we modify the dynamics of S described in Section 3.1 as follows

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t + S_{t-} \int_{\mathbb{R}} (\mathrm{e}^z - 1) \widetilde{N}(\mathrm{d}t, \mathrm{d}z),$$

where W is a Brownian motion and $\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$ is a compensated Poisson random measure. Once again, we do not specify a model for the volatility process σ. We assume only that σ is a right-continuous $\mathbb{F}$-adapted process that involves independently of W and N. Under certain growth and smoothness restrictions on the payoff function φ we are able to derive unique prices for claims with payoffs of the form $\varphi(X_T, [X]_T)$ relative to the T-maturity calls and puts, where $[X]$ denotes the quadratic variation of X. Additionally, when X experiences only a finite number of jump sizes (i.e., $\nu(\mathrm{d}z) = \sum_i \delta_{z_i}(z)\mathrm{d}z$), we are also able to find hedging portfolios that replicate the claim payoff with probability one.

3.3 Variance Swaps on Time-Changed Markov Processes

For an asset $S = \mathrm{e}^X$, the floating leg of a variance swap (VS) pays $[X]_T$ to the long side at time T. Variance swaps are among the most liquid and widely traded path-dependent claims. Indeed, VS contracts on stock indices now have bid-offer spreads narrower than those of at-the-money options. It is therefore important to robustly determine the fair-strike $\mathbb{E}[X]_T$. When the underlying S has continuous sample paths, [Neu90] shows that the fair strike of a VS is equal to the value of two European log contracts $\mathbb{E}[X]_T = -2\mathbb{E}(X_T - X_0)$. However, for certain assets, the no-jump assumption is restrictive. With this in mind, in [CLL17b], we model the log price X of an underlying as a general time-changed Markov process

$$X_t = Y_{\tau_t}, \qquad \mathrm{d}Y_t = b(Y_t)\mathrm{d}t + a(Y_t)\mathrm{d}W_t + \int_{\mathbb{R}} z \widetilde{N}(Y_{t-}, \mathrm{d}t, \mathrm{d}z),$$

where b is chosen so that $S = \mathrm{e}^X$ is a martingale, τ is a continuous stochastic clock, W is a Brownian motion and $\widetilde{N}(y, \mathrm{d}t, \mathrm{d}z) = N(y, \mathrm{d}t, \mathrm{d}z) - \nu(y, \mathrm{d}z)\mathrm{d}t$ is a compensated Poisson random measure with a state-dependent Lévy kernel ν. We do not specify a model for τ; it may be any continuous, strictly increasing stochastic process and it my have arbitrary correlation with W and N. In this setting, we show that the fair strike of a VS satisfies

$$\mathbb{E}[X]_T = \mathbb{E}G(X_T) - G(X_0), \tag{10}$$

where the function G is any solution of the integro-differential equation

$$\mathcal{A}G(y) = a^2(y) + \int_{\mathbb{R}} z^2 \nu(y, \mathrm{d}z), \qquad \mathcal{A} = \tfrac{1}{2}a^2(y)(\partial_y^2 - \partial_y) + \int_{\mathbb{R}} \nu(y, \mathrm{d}z) \left(\theta_z - 1 - (\mathrm{e}^z - 1)\,\partial_y\right), \tag{11}$$

with $\theta_z G(y) := G(y + z)$. For certain choices of Markov processes Y, we are able to solve (11) explicitly. It is important to note that the function G does not depend on the time-change τ. Thus, using (9) and (10), we are able to value the fair strike of a VS relative to European calls and puts with no risk of misspecifying the stochastic clock τ.

⋆ ⋆ ⋆

I have employed semiparametric methods in two other papers. In [LLCMA16], we consider a Lévy subordinated diffusion model for a risky asset S and we use options data to extract the drift and Lévy measure of the subordinator. And in [LL16], we provide a model-free approach to optimally statically hedge a variety of options.

Preprints versions of most of my papers are available for download at: http://arxiv.org/a/lorig_m_1.html.

Links to published version of my papers can be found on my website: https://mattlorig.yolasite.com/.

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