The Exact Implied Volatility Smile for Exponential Lévy Models

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Abstract

For any exponential Lévy model whose diffusion component is nonzero, we provide an exact series representation for the implied volatility of a European call option. Numerical examples are provided.

Keywords: Implied Volatility, Exponential Lévy.

1 Introduction

Various approaches have been taken to studying the implied volatility surface induced by a given class of equity models. Most of these approaches explore asymptotic regimes of strikes and maturities (long expiries, short expiries, large strikes, small strikes, etc.) or a specific feature of the implied volatility surface, such as the at-the-money skew. An exhaustive review of the implied volatility literature would be prohibitive. But, we mention a few papers that deal with exponential Lévy processes in particular. The short maturity volatility smile is studied in Figueroa-López and Forde (2012), and the long maturity smile in Figueroa-López, Forde, and Jacquier (2011). The model-free results of Lee (2004); Gao and Lee (2011) take a particularly simple form for exponential Lévy models and, as such, are useful for studying extreme strike behavior (large and small) of implied volatility. For a review of results on asymptotics for implied volatilities in exponential Lévy models we refer the reader to Tankov (2011); Andersen and Lipton (2012).

Our approach to studying implied volatility is quite different from the above-mentioned works. Rather than exploit a particular maturity and/or strike regime, we exploit the simple structure of exponential Lévy models. In doing so, we obtain an exact formula (written as an infinite series) for the implied volatility of a given call option. As far as we are aware, this is the first time a formula for the exact implied volatility has been given in any framework – exponential Lévy or otherwise. We also mention that our formula is extremely

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simple to derive. While previous authors have used advanced mathematical techniques (e.g., saddle-point methods, moment analysis, large-deviation principle, etc.) to derive asymptotic implied volatility results, our exact result requires only basic calculus.

The rest of this paper proceeds as follows. In section 2 we introduce the class of exponential Lévy models. In section 3, we review how European options may be valued in an exponential Lévy setting using generalized Fourier transforms. Finally, in section 4 we define implied volatility and – for a given call option and exponential Lévy model – derive a formula for the corresponding implied volatility. The main result of our work is summarized in Theorem 7. Numerical examples are provided at the conclusion of this paper.

2 Exponential Lévy Models

In this section we review the class of exponential Lévy models. A detailed development can be found in Cont and Tankov (2004); Øksendal and Sulem (2005). We assume a frictionless market, no arbitrage and take an equivalent martingale measure $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$. All processes defined below live on this space. Let $S$ represent the price process of a risky asset. The main assumptions of this paper is that $S$ can be modeled as an exponential Lévy process

$$S_t = e^{X_t}, \quad dX_t = \gamma \, dt + \sigma \, dW_t + \int_{|z|<R} z \, d\tilde{N}_t(dz) + \int_{|z|\geq R} z \, dN_t(dz), \quad X_0 = x.$$ 

Here, $R \in [0, \infty]$, the volatility satisfies $\sigma > 0$, $W$ is a Brownian motion and $N$ is a Poisson random measure characterized by Lévy measure $\nu$

$$\mathbb{E} N_t(dz) = \nu(dz) \, dt, \quad d\tilde{N}_t(dz) = dN_t(dz) - \nu(dz) \, dt.$$ 

We require that $\nu$ satisfy

$$\int_{\mathbb{R}} \min(1, z^2) \nu(dz) < \infty, \quad \int_{|z| \geq R} e^z \nu(dz) < \infty. \quad (1)$$

The first condition must be satisfied by all Lévy measures. The second condition guarantees that $\mathbb{E} S_t < \infty$ for all $t \in \mathbb{R}^+$. Valid choices for $R$ depend on the Lévy measure $\nu$. We can always choose $R = 1$. If $\int_{|z| \geq 1} |z| \nu(dz) < \infty$ then we may choose $R = \infty$. For simplicity, we assume $S$ pays no dividends and the risk-free rate of interest is zero. Thus, $S$ must be a martingale. The martingale condition is satisfied if and only if

$$\gamma = -\frac{1}{2} \sigma^2 - \int_{\mathbb{R}} \nu(dz) \left( e^z - 1 - z \mathbb{1}_{|z|<R} \right). \quad (2)$$
3 European Option Pricing

We consider a European option expiring at time $t > 0$ with payoff $h(X_t)$. Using risk-neutral pricing, the time-zero value of such an option is the $P$-expectation of the option payoff

$$u(t, x) = E_x h(X_t).$$

Lewis (2001); Lipton (2002) independently show that $u(t, x)$ can be computed using generalized Fourier transforms. We review their method below. For brevity, we do not include any proofs. Let $\phi(\lambda)$ denote the characteristic exponent of $X$

$$\phi(\lambda) := \log E e^{i\lambda X_1}, \quad \phi(\lambda) = i\gamma\lambda - \frac{\sigma^2}{2} \lambda^2 + \int_{\mathbb{R}} \nu(dz) \left( e^{i\lambda z} - 1 - i\lambda z \mathbb{1}_{|z| < R} \right).$$

We assume that $\phi$ is analytic in an infinite strip $\Lambda^\phi$ of the complex plane

$$\Lambda^\phi := \{ \lambda \in \mathbb{C} : \text{Im}(\lambda) \in (\lambda^-_\phi, \lambda^+_\phi) \},$$

$$\lambda^-_\phi = \inf \left\{ \lambda < 0 : \int_{-\infty}^{-1} \nu(dz) e^{\lambda z} < \infty \right\}, \quad \lambda^+_\phi = \sup \left\{ \lambda > 1 : \int_{1}^{\infty} \nu(dz) e^{\lambda z} < \infty \right\}.$$ 

Let $\hat{h}(\lambda)$ denote the generalized Fourier transform of $h(x)$

$$\hat{h}(\lambda) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{-i\lambda x} h(x).$$

We assume $\hat{h}(\lambda)$ is analytic on an infinite strip of the complex plane of the form $\Lambda^h := \{ \lambda \in \Lambda^\phi : \text{Im}(\lambda) \in (\lambda^-_h, \lambda^+_h) \}$. Let $\lambda = \lambda_r + i\lambda_i$ where $\lambda_r, \lambda_i \in \mathbb{R}$ and fix the imaginary component: $\lambda_i \in (\lambda^-_h, \lambda^+_h)$. Then the value of the option $u(t, x)$ is given by

$$u(t, x) = \int_{\mathbb{R}} d\lambda_r e^{t\phi(\lambda)} \hat{h}(\lambda) \psi_\lambda(x), \quad \psi_\lambda(x) = \frac{1}{\sqrt{2\pi}} e^{i\lambda x}.$$ 

4 Implied Volatility

In this section we fix $(t, x)$ and a call option payoff $h(x) = (e^x - e^k)^+$. Note that

$$\hat{h}(\lambda) = \frac{-e^{-k-i\lambda}}{\sqrt{2\pi} (i\lambda + \lambda^2)}, \quad \text{Im}(\lambda) < -1.$$ 

We also fix $\sigma > 0$ and a Lévy measure $\nu = \epsilon \mu$ where $\epsilon \geq 0$ and $\mu$ is any Lévy measure that satisfies (1). By (2), the parameter $\gamma$ is fixed by $\sigma$ and $\nu$. To keep track of their dependence on $\epsilon$ we write the characteristic
exponent as $\phi(\lambda)$ and the option price $u(t, x)$ as $\phi^x(\lambda)$ and $u^x(t, x)$ respectively. We have

$$
\phi^x(\lambda) = \phi_0(\lambda) + \varepsilon \phi_1(\lambda),
$$

$$
\phi_0(\lambda) = \frac{1}{2} \sigma^2 (-\lambda^2 - i \lambda),
$$

$$
\phi_1(\lambda) = -i \lambda \int_{\mathbb{R}} \mu(dz) (e^{\varepsilon z} - 1 - \varepsilon \mathbb{1}_{\{|z| < R\}}) + \int_{\mathbb{R}} \mu(dz) (e^{i \lambda z} - 1 - i \lambda z \mathbb{1}_{\{|z| < R\}}),
$$

and

$$
u^x = \int_{\mathbb{R}} d\lambda e^{t \phi^x(\lambda)} \hat{h}(\lambda) \psi_\lambda. \tag{3}
$$

To ease notation, we have dropped the subscript $r$ from $d\lambda_r$. The following definitions will be useful:

**Definition 1.** The Black-Scholes Price $u^{BS} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$
u^{BS}(\rho) := \int d\lambda e^{t \phi^{BS}(\lambda; \rho)} \hat{h}(\lambda) \psi_\lambda, \quad \phi^{BS}(\lambda; \rho) = \frac{1}{2} \sigma^2 (-\lambda^2 - i \lambda).$$

**Definition 2.** The Implied Volatility is defined implicitly as the unique number $\sigma^x \in \mathbb{R}^+$ such that

$$
u^{BS}(\sigma^x) = u^x,$$  

where $u^x$ is given by (3).

**Remark 3.** For $0 < t < \infty$ the existence and uniqueness of the implied volatility $\sigma^x$ can be deduced by using the general arbitrage bounds for call prices and the monotonicity of $u^{BS}$.

**Remark 4.** Note that $u^{BS}$ is an invertible analytic function that satisfies $\partial_\rho u^{BS}(\rho) > 0$ for all $\rho > 0$. By the Lagrange inversion theorem, the inverse $[u^{BS}]^{-1}$ of such a function is also analytic.

Our goal is to find an explicit formula for the implied volatility $\sigma^x$. To this end, we note that

$$
\exp(t \phi^x(\lambda)) = \exp(t(\phi_0(\lambda) + \varepsilon \phi_1(\lambda))) = \exp(t \phi_0(\lambda)) \sum_{n=0}^{\infty} \frac{1}{n!} (t \varepsilon \phi_1(\lambda))^n. \tag{5}
$$

Inserting (5) into (3) we obtain the following series representation $^1$ for $u^x$

$$
u^x = \sum_{n=0}^{\infty} \varepsilon^n u_n, \quad u_n = \frac{t^n}{n!} \int_{\mathbb{R}} d\lambda \exp(t \phi_0(\lambda)) (\phi_1(\lambda))^n \hat{h}(\lambda) \psi_\lambda. \tag{6}
$$

Note in particular that $u_0 = u^{BS}(\sigma)$.

From (6), it is clear that $u^x$ is an analytic function of $\varepsilon$. It is a useful fact that the composition of two analytic functions is also analytic (see Brown and Churchill (1996), section 24, p. 74). Thus, in light

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$^1$By Fubini’s Theorem, exchanging the order of summation and integration is allowed since $\int_{\mathbb{R}} d\lambda |\exp(t \phi^x(\lambda)) \hat{h}(\lambda) \psi_\lambda| < \infty$. 

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of Remark 4, we deduce that $\sigma^\varepsilon = [u^{BS}]^{-1}(u^\varepsilon)$ is an analytic function and therefore has a power series expansion in $\varepsilon$. We write this expansion as follows

$$\sigma^\varepsilon = \sigma_0 + \delta^\varepsilon, \quad \delta^\varepsilon = \sum_{k=1}^{\infty} \varepsilon^k \sigma_k.$$  \hfill (7)

Taylor expanding $u^{BS}$ about the point $\sigma_0$ we have

$$u^{BS}(\sigma^\varepsilon) = u^{BS}(\sigma_0 + \delta^\varepsilon)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (\delta^\varepsilon \partial_\sigma)^n u^{BS}(\sigma_0)$$

$$= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{k=1}^{\infty} \varepsilon^k \sigma_k \right)^n \partial_\sigma^n u^{BS}(\sigma_0)$$

$$= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{k=1}^{n} \prod_{j=1}^{k} \sigma_j \right) \varepsilon^k \partial_\sigma^n u^{BS}(\sigma_0)$$

$$= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[ \sigma_k \partial_\sigma + \sum_{n=2}^{\infty} \frac{1}{n!} \left( \sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_j \right) \partial_\sigma^n u^{BS}(\sigma_0) \right] u^{BS}(\sigma_0). \hfill (8)$$

Now, we insert expansions (6) and (8) into (4) and collect terms of like order in $\varepsilon$

$$\mathcal{O}(1) : \quad u_0 = u^{BS}(\sigma_0),$$

$$\mathcal{O}(\varepsilon^k) : \quad u_k = \sigma_k \partial_\sigma u^{BS}(\sigma_0) + \sum_{n=2}^{\infty} \frac{1}{n!} \left( \sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_j \right) \partial_\sigma^n u^{BS}(\sigma_0), \quad k \geq 1.$$ 

Solving the above equations for $\{\sigma_k\}_{k=0}^{\infty}$ we find

$$\mathcal{O}(1) : \quad \sigma_0 = \sigma,$$

$$\mathcal{O}(\varepsilon^k) : \quad \sigma_k = \frac{1}{\partial_\sigma u^{BS}(\sigma)} \left( u_k - \sum_{n=2}^{\infty} \frac{1}{n!} \left( \sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_j \right) \partial_\sigma^n u^{BS}(\sigma) \right), \quad k \geq 1. \hfill (9)$$

**Remark 5.** The right hand side of (9) involves only $\sigma_j$ for $j \leq k - 1$. Thus, the $\{\sigma_k\}_{k=1}^{\infty}$ can be found recursively.

**Remark 6.** Note that $\partial_\sigma^n u^{BS}(\sigma)$ is easily computed using

$$\partial_\sigma^n u^{BS}(\sigma) = \int d\lambda \left( \partial_\sigma^n e^{\phi_0(\lambda)} \right) \tilde{h}(\lambda) \psi_\lambda.$$
Explicitly, up to $O(\varepsilon^4)$ we have

\begin{align*}
O(\varepsilon) & : \quad \sigma_1 = \frac{u_1}{\partial_{u_0}}, \\
O(\varepsilon^2) & : \quad \sigma_2 = \frac{u_2 - \frac{1}{2} \sigma_1^2 \partial_{u_0}^3 u_0}{\partial_{u_0}} , \\
O(\varepsilon^3) & : \quad \sigma_3 = \frac{u_3 - (\sigma_2 \sigma_1 \partial_{u_0}^2 + \frac{1}{3} \sigma_1^3 \partial_{u_0}^3) u_0}{\partial_{u_0}} , \\
O(\varepsilon^4) & : \quad \sigma_4 = \frac{u_4 - (\sigma_3 \sigma_1 \partial_{u_0}^2 + \frac{1}{2} \sigma_2^2 \partial_{u_0}^2 + \frac{1}{2} \sigma_2 \sigma_1 \partial_{u_0}^3 + \frac{1}{4} \sigma_1^4 \partial_{u_0}^4) u_0}{\partial_{u_0}} .
\end{align*}

We summarize our main result in the following theorem:

**Theorem 7.** The implied volatility $\sigma^\varepsilon$ defined in (4) is given explicitly by (7) where $\sigma_0 = \sigma$ and \{\(\sigma_k\)\}_{k=1}^\infty are given by (9).

**Remark 8.** We emphasize: we have made no assumption about the size of $\varepsilon$. Theorem 7 is valid for any $\varepsilon \geq 0$. In particular, one can always choose $\varepsilon = 1$.

**Remark 9.** Everything we have done so far is exact. The accuracy of the implied volatility expansion (7) is limited only by the number of terms one wishes to compute.

Define the $O(\varepsilon^n)$ approximation of the implied volatility

$$\sigma^{(n)} := \sum_{k=0}^{n} \varepsilon^k \sigma_k.$$ 

At the end of this document, we provide numerical examples illustrating convergence of $\sigma^{(n)}$ to $\sigma^\varepsilon$ for three well-known exponential Lévy models:

- the Jump-diffusion model of Merton (1976): figure 1,
- the Variance Gamma model of Madan, Carr, and Chang (1998): figure 2,
- the CGMY model of Carr, Geman, Madan, and Yor (2002): figure 3.

We plot implied volatility as a function of the log-moneyness to maturity ratio, LMMR := \((k-x)/t\). In all three models, we see excellent convergence of $\sigma^{(n)}$ to $\sigma^\varepsilon$. Convergence is fastest for values of $k$ near $x$ and slows as $k$ moves away from $x$.

**Remark 10.** Although our focus has been on exponential Lévy models, the exact implied volatility expansion outlined above will work for any model whose European call price can be expanded analytically in $\varepsilon$ as

$$u^\varepsilon = u^{BS} + \sum_{k=1}^{\infty} \varepsilon^k u_k ,$$

where $\varepsilon$ is some model-specific parameter. See, for example, Lorig (2012b,a).
5 Acknowledgments

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References


Figure 1: Using the Merton model, we plot $\sigma(n)$ (solid blue) and $\sigma^\varepsilon$ (dashed black) as a function of LMMR. The following parameters are used throughout: $s = 0.15$, $m = -0.15$, $\sigma = 0.35$, $\varepsilon = 0.75$, $t = 0.33$.

Merton model:

\[
\nu(dz) = \frac{\varepsilon}{\sqrt{2\pi s^2}} \exp\left(\frac{-(z-m)^2}{2s^2}\right) dz.
\]
Figure 2: Using the Variance Gamma model, we plot $\sigma^{(n)}$ (solid blue) and $\sigma^\varepsilon$ (dashed black) as a function of LMMR. The following parameters are used throughout: $G = 1.0$, $M = 3.0$, $\sigma = 0.35$, $\varepsilon = 0.3$, $t = 0.15$.

Variance Gamma model:  
$$
\nu(dz) = \varepsilon \left( \frac{e^{Gz}}{-z} \mathbb{I}_{\{z<0\}} + \frac{e^{-Mz}}{z} \mathbb{I}_{\{z>0\}} \right) dz.
$$
Figure 3: Using the CGMY model, we plot $\sigma^{(n)}$ (solid blue) and $\sigma^\varepsilon$ (dashed black) as a function of LMMR. The following parameters are used throughout: $G = 2.0$, $M = 4.0$, $Y = -3.0$, $\sigma = 0.35$, $\varepsilon = 0.3$, $t = 0.5$.

**CGMY model:**

$$\nu(dz) = \varepsilon \left( \frac{e^{Gz}}{|z|^{1+Y}1_{\{z<0\}}} + \frac{e^{-Mz}}{z^{1+Y}1_{\{z>0\}}} \right) dz.$$