Elastic Solutions for Eccentrically Loaded, Slender, Precast and Prestressed Concrete Spandrel Beams

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Abstract:

Spandrel beams in precast concrete buildings are widely used to support double-tee deck beams. Spandrel beams of deep cross sections, resisting eccentric loads from double-tee beams, can be susceptible to excessive lateral deformations and serviceability failures before reaching their strength limits. In this paper, exact and approximate analytical solutions are derived from second order elastic analysis to estimate maximum lateral deflections in laterally restrained and unrestrained spandrel beams under eccentric and uniformly distributed loads. Continuous lateral support is provided at the elevation of the floor deck in restrained spandrel beams. An equivalent loading method is proposed to obtain the approximate analytical solutions, in which the differential equations of equilibrium governing the problem are simplified by modifying the actual loads in spandrel beams. Numerical solutions are also obtained from three-dimensional finite element analyses and their results are found to be in close agreement with those of analytical solutions.

CE Database subject headings:

Precast concrete; Prestressed concrete; Lateral displacement; Eccentric loads; Analytical techniques; Numerical analysis.
Introduction

Precast and prestressed concrete spandrel beams (Fig. 1) are usually used in the perimeter of precast concrete buildings and support precast double-tees serving as deck beams. There are three common types of spandrel beams depending upon the manner in which the double tees are supported; spot-corbel spandrels (Fig. 1.a) with discontinuous ledges, L-shaped spandrels with continuous ledges (Fig. 1.b) and pocket spandrels (Fig. 1.c) with rectangular cutouts. The spandrel beam resists eccentric loading which stems from vertical double-tee loads acting through the horizontal offsets provided by corbels or cut-outs. The spandrel beam is supported simply at its soffits by the column corbels. Torsional constraint, which is an additional connection of the spandrel to the columns prevents twisting of the spandrel ends, is achieved by means of threaded inserts (i.e., tie-back bolts) at two elevations above the column corbels. The load bearing spandrel beam is also laterally restrained along the mid-height level of its web by spandrel-to-double-tee connections (i.e., deck ties).

Previous research

There are several publications on the study of the interaction of and design for combined shear, torsion and bending in reinforced or prestressed concrete flexural members (Raths 1984; Klein 1986; Lucier et al. 2007). However, the vast majority of this work has focused on stress and strength design considerations, and mention is seldom made of lateral deflection and serviceability of flexural members. Only a limited amount of research on the stability of tall, slender ‘reinforced’ concrete beams is found in the
technical literature, including experimental studies from the 1950s and 1960s conducted in the U.S. (Hansel and Winter 1959; Sant and Bletzacker 1961; Marshall 1969) and more recent ones (Revathi and Mennon 2006; Revathi and Mennon 2007; Kalkan 2009). However, all of these studies aimed at exploring the lateral instability of concentrically loaded reinforced concrete beams. Historically, lateral deflections have often been ignored in the design of precast and prestressed concrete spandrel beams, as well as many other concrete members, since they are assumed to be too stocky for lateral deflections to be more critical than stress or force considerations. However, due to technological improvements and industry needs, longer (up to 60 ft) and thinner (as little as 8 in.) spandrel beams are being introduced to the market. Based on this current trend, lateral deflections for slender spandrel beams need to be evaluated.

In order to find the analytical solution for maximum lateral deflections in spandrel beams, the governing differential equations of equilibrium need to be established first. To this end, the differential equations for the lateral-torsional buckling problem in concentrically loaded elastic beams, which are homogenous, isotropic and prismatic, can be used. The solution reported by Timoshenko in 1936 (Timoshenko and Gere 1961) is well known and referenced in many textbooks on steel design (e.g., Bleich 1952; Gaylord et al. 1992; Salmon and Johnson 1996; Galambos et al. 1996). The differential equations of equilibrium can be easily extended to include torsional effects and solved to obtain expressions for lateral deflection and angle of twist.

Linearly elastic material behavior can be assumed only for prestressed concrete, since the actual response of pure concrete is highly nonlinear. Prestress in concrete usually
eliminates or significantly reduces tensile stresses in the members under service loads, thus providing resistance to cracking. Concrete also exhibits nonlinear stress-strain relationship for compressive stresses greater than about one-half of its compressive strength. However, a reduced modulus of elasticity $E_r$ can be used to represent the concrete behavior for deflection calculations. Such an approach is often used in practice, for example, Sant and Bletzacker (1961) recommend a value for the reduced modulus $E_r$ equal to $0.687E_c$ to account for nonlinearity in the stress-strain relation for concrete in compression in tall, slender reinforced concrete beams.

In this work, second-order elastic analyses were conducted to evaluate lateral deflections and angle of twist in restrained and unrestrained rectangular beams subjected to eccentric vertical loading. Finding the closed-form solution to the differential equations governing this problem turned out impossible. Therefore, an equivalent loading procedure was proposed, in which governing differential equations were significantly simplified and solved to obtain approximate expressions for deflection and angle-of-twist. Finally, analytical results were compared with those obtained from the nonlinear finite element analyses of the spandrel beams.

**Lateral restraints with deck-tie plates**

Load bearing spandrel beams typically have deck-ties along its span length to provide a lateral connection between spandrel and double tees. However, some practicing engineers do not rely on the deck-ties as providing sufficient lateral supports for spandrel beams. There are valid reasons for this; first, the ties are not proportioned to meet a
design goal; second, deck-tie stress/force not computed; third, the influence of cyclic loading not considered on fatigue life; fourth, protection from corrosion not provided; so the reliability of these connections is not known to provide a lateral connection between spandrel and double-tees.

A typical deck-tie connection, as illustrated in Fig. 2, is built on site by simply welding a steel plate to two other steel elements (e.g., plates or angles) which were already embedded into the spandrel web and the double-tee flange in the precast plant. Deck-ties connecting the spandrel web to the double-tees are often placed near the neutral axis elevation of the spandrel section, in which case they do not restrain twisting deformations in a significant manner. However, once these connections are established, the pattern of lateral deflections changes since the beam cannot deflect laterally at the level of the double tee connections as long as the deck-tie welds are intact. Yet, the welding of the deck-tie plates does not take place immediately upon placing the double tees. So, it is somewhat risky to rely on these connections while the structure is being erected. Moreover, high-cycle fatigue (from vehicle loading), low-cycle fatigue (from extreme loading), stresses from volumetric effects (temperature, creep and shrinkage of concrete) and corrosion raise further doubts on the reliability of the deck-tie connections. Therefore, the lateral deflection of a spandrel beam should be investigated for two idealized cases (Fig. 3); (a) unrestrained beam and (b) restrained beam, in which the lateral deformations are prevented by deck-ties at the mid-height level along the span length.
In extreme cases of large twisting angles, the supports of the double tees can be lost, leading to the collapse of the structure. Fig. 3 shows the deformed shapes of spandrel webs of restrained and unrestrained beams. If the lateral deflection at the bottom of the restrained beam is larger than the overlap distance at the bearing pad (Fig. 2), the double-tee beams will slip out of the spandrel ledge and collapse. There is no such collapse mechanism for laterally unrestrained beams since lateral deformations always tend to move toward the double-tees. Still, unrestrained beams can experience large lateral deflections before reaching their design capacities and fail to satisfy serviceability requirements.

**Laterally unrestrained and concentrically loaded beams**

In general, slender elastic beams with deep and narrow rectangular cross sections do not experience lateral deflections when concentric loads (i.e., without lateral eccentricity producing torsion) are applied. At a critical load (buckling load), slender beams suddenly undergo large lateral deformations, which is called lateral and torsional buckling. This type of instability likely occurs when the flexural rigidity of a beam in the plane of bending is significantly larger than that in the out-of-plane direction. The theory of lateral-torsional buckling of slender beams under concentric loads is well known and extensively investigated for steel beams (e.g., Bleich 1952; Gaylord et al. 1992; Salmon and Johnson 1996; Galambos et al. 1996). Before looking into the behavior of eccentrically loaded beams, a short discussion on concentrically loaded beams is necessary for several reasons; (a) differential equations of equilibrium derived for a
concentrically loaded beam can be easily extended to the case of an eccentrically loaded beams (b) the use of the buckling load parameter in the derivations will significantly simplify the analytical expressions, and (c) the influence of eccentric loading on the lateral response will be more meaningful when it is compared with the response of a concentrically loaded beam.

A deep and narrow rectangular beam subjected to strong-axis bending-moment couples $M_o$ at the ends suddenly becomes unstable, deforms laterally and twists, as depicted in Fig. 4.a, when the moment $M_o$ attains a critical value $M_{cr}$. In order to determine the critical buckling moment, governing equilibrium equations are established for a slightly deformed shape (or buckled shape) of the beam. The smallest value of bending moment couples $M_o$ that maintains the equilibrium of the beam in such buckled shape is equal to the critical value of buckling moment $M_{cr}$.

The global and local coordinate axes, $x$-$y$-$z$ and $x^*$-$y^*$-$z^*$, respectively, are selected, as shown in Fig. 4.a. The vertical deformation of the centroid of the beam cross section $v$, the lateral deformation $u$, and the angle of twist $\phi$ are defined in the buckled shape of the beam, as illustrated in Fig. 5. The angle of twist $\phi$ is assumed to be positive when the cross section rotates from the $x$- to the $y$-axis, or when the rotation vector defined using ‘right-hand rule’ is parallel to the positive direction of $z$. It should be noted that the deformations, $u$, $v$, and $\phi$, are very small and higher-order terms will be neglected. The governing differential equations of equilibrium are to be established in the buckled shape of the beam. Therefore, the curvatures of the centerline of the beam in $y^*$-$z^*$- and $x^*$-$z^*$- planes and the moments $M_{x^*}$, $M_{y^*}$, and $M_{z^*}$, are required. The positive directions of
internal moments are shown in Fig. 4.b. Since the angle of twist is very small, the
curvatures in the \( y^*z^* \) and \( x^*z^* \) planes are equal to the curvatures in the \( xz \) and \( yz \)-
planes (i.e., \( d^2u/dz^2 \) and \( d^2v/dz^2 \), respectively). Likewise, only the dominated terms of the
components of \( M_x \) (or \( M_o \)) in the local coordinate axes, \( x^*-y^*-z^* \), are retained in the
transformation from \( x-y \) to \( x^*-y^* \) as seen in Fig. 6. This leads to the differential equations
of equilibrium in the form:

\[
\begin{align*}
-EI_x y'' &= M_{x^*} \cong M_o \quad (1) \\
EI_y u'' &= M_{y^*} \cong -\phi M_o \quad (2) \\
GJ\phi' &= M_{z^*} \cong u'M_o \quad (3)
\end{align*}
\]

where \( E \) is the modulus of elasticity; \( I_x \) is the moment of inertia of the beam section
about the strong-axis of bending; \( I_y \) is the moment of inertia about the weak-axis; \( G \) is the
shear modulus of elasticity; \( J \) is the torsional constant for the beam section. The warping
torsional resistance is negligible for beams with deep and narrow rectangular cross-
sections and so the warping effect does not appear in Eq. (3). It should be noted that
lateral torsional buckling involves torsion even though there is no torsional loading. The
internal torsional moment arises as a component of the deformation pattern and hence Eq.
(3) is necessary. After simplification, a second-order linear differential equation for the
angle of twist \( \phi \) with constant coefficient can be obtained. The solution of this
differential equation gives the critical buckling value (\( M_{cr} \)) of the end moments \( M_o \):

\[
M_{cr} = \frac{\pi^2}{L^2} \sqrt{EI_y GJ}
\]
Eq. (4) is derived for the loading case of equal and opposite end moments. For cases with moment gradient, the critical moment can be adjusted using the bending coefficient $C_b$, based on approximate expressions 0-0. The coefficient $C_b$ is equal to unity for uniform bending (i.e., equal end moments) and 1.13 for uniformly distributed vertical loads. Incorporating the $C_b$ factor, the critical moment given by Eq. (4) becomes:

$$M_{cr} = C_b \frac{\pi}{L} \sqrt{EI \gamma} \frac{GJ}{L}$$

(5)

**Laterally unrestrained and eccentrically loaded beams**

In order to investigate the influence of eccentric loading on the lateral deflection of spandrel beams, a simplified structural system (Fig. 7.a), in which a slender beam with rectangular cross section is subjected to uniformly distributed load $q$ with an eccentricity $e$, is taken into account. The bending moment $M_x$ has a parabolic distribution along the span length, $M_x = (qL^2/2)[q/L-(q/L)^2]$. Due to the torsionally fixed supports, as well as the symmetry of the beam and loading pattern, the torsional reactions at the ends ($M_T = qe L/2$) are equal, but in opposite directions. The distribution of internal torsion along the length of beam is a linear function of $z$, $M_z = M_T(1-2z/L)$, with the left end of the beam undergoing positive torsion.

Timoshenko’s solution does not explicitly address beams with torsional loading (i.e., lateral eccentricity), but the formulation of the governing differential equations can be easily extended to include torsional loading (Schultz and Magaña 2000). The components of the torsion along the $x^*$, $y^*$, and $z^*$ axes in the deformed position must be
included in the three moment equilibrium equations, Eqs (1), (2), and (3). As illustrated in Fig. 7.b, a positive beam slope \(du/dz\) in the horizontal \(x-z\) plane produces a torsional component \(M_z du/dz\) in the local \(x^*\) axis which is negative according to the sign convention described in Fig. 4.b. Thus, positive torsion \(M_z\) acting through a positive horizontal slope \(du/dz\) generates a negative moment contribution for \(x\)-axis equilibrium in Eq. (1). Similarly, in Fig. 7.b, a positive beam slope \(dv/dz\) in the vertical \(y-z\) plane produces a torsional component \(M_z dv/dz\) in the local \(y^*\) axis which opposes positive bending as indicated in Fig. 4. Thus, positive torsion \(M_z\) acting through a positive vertical slope \(dv/dz\) produces a negative moment contribution for \(y\)-axis equilibrium in Eq. (2).

\[
\begin{align*}
- EI_x y'' &= M_x - u'M_z \\
EI_y u'' &= - \phi M_x - v'M_z \\
GJ\phi' &= u'M_x + M_z
\end{align*}
\]  

(6)  

(7)  

(8)

The case of a laterally unrestrained beam under eccentric loading introduces a parabolic function of length \(z\) for bending moment \(M_x\) and a linear function of \(z\) for torsion \(M_z\), which makes coefficients of Eqs (6), (7), and (8) variable. As a result, the closed-form solution of Eqs (6), (7), and (8) becomes difficult, if not impossible. Still, an approximate closed-form solution can be found if the actual loading in the beam is replaced with an equivalent loading, shown in Fig. 8. The actual loading case consists of uniformly distributed load \(q\) acting over an eccentricity \(e\). For the equivalent loading approach, fictitious but approximately equivalent effects are defined, as seen in Fig. 8. The actual load is replaced by the end moments \(M_o (= qoL^2/8)\) and a uniformly distributed torque \(m_o\) modified by the load parameter \(q_o\), \(q_o=q/C_b\) where \(C_b\) is the bending
The moment $M_o$ has been assumed so as to obtain the same maximum bending moment for both the actual and equivalent loading conditions. Such equivalent loads yield a differential equation of equilibrium with constant coefficient (as presented later) and provide an approximate closed-form solution. The accuracy of the equivalent loading approach will be discussed in detail later by comparing numerical results from actual loading with analytical results obtained from the equivalent loading.

Differential equations can be simplified by neglecting the last terms of Eqs (6) and (7), $(-M_z du/dz$ and $-M_z dv/dz)$, due to the fact that the torsional moment $M_z$ is generally very small relative to $M_o$. Under this condition, for the case of laterally unrestrained and eccentrically loaded beam (with the equivalent loading), the differential equations of equilibrium are:

$$-EI_y v'' = M_o$$  \hspace{1cm} (9)

$$EI_y u'' = -\phi M_o$$  \hspace{1cm} (10)

$$GJ\phi' = u'M_o + \frac{eq_L}{2} \left(1 - \frac{2z}{L}\right)$$ \hspace{1cm} (11)

Solving Eq. (10) for $\phi$ and substituting its derivative $d\phi/dz$ into Eq. (11), we find

$$u'' + u' \frac{(M_o)^2}{EI_y GJ} = -\frac{eq_L}{2} \left(1 - \frac{2z}{L}\right) \frac{M_o}{EI_y GJ}$$ \hspace{1cm} (12)

Introducing (of Eq. (4))

$$M_{cr} = \frac{\pi}{8} EI_y GJ, \quad M_o = \frac{q_s L^2}{8}, \quad \text{and} \quad \alpha = \frac{M_o \pi}{M_{cr} L}$$ \hspace{1cm} (13)

Eq. (12) is transformed into
\[ u'' + \alpha^2 u' = -4\alpha^2 \frac{e}{L} + 8\alpha^2 \frac{e}{L} \frac{z}{L} \]  

The solution of equation (14) gives the following formula for the lateral deflection \( u \) as a function of the position \( z \) along the beam.

\[
 u(z) = A_i \cos(\alpha z) + A_j \sin(\alpha z) + A_3 + 4 \frac{e}{L} \left( \frac{z^2}{L} - z \right) \]  

(15)

The constants, \( A_1, A_2 \) and \( A_3 \), are obtained using the following boundary conditions;

\( u(0) = \phi(0) = 0 \) and \( u'(L/2) = 0 \).

Considering that

\[
 \phi = -\frac{E I_y}{M_o} u'' = \frac{E I_y}{M_o} \left( A_i \alpha^2 \cos(\alpha z) + A_j \alpha^2 \sin(\alpha z) - 8 \frac{e}{L^2} \right) 
\]

(16)

the constants, \( A_1, A_2 \) and \( A_3 \), turn out to be:

\[
 z = 0, \; \phi = 0, \; \frac{E I_y}{M_o} \left( A_i \alpha^2 + 0 - 8 \frac{e}{L^2} \right) = 0, \; A_i = -8 \frac{e}{\alpha^2 L^2} 
\]

(17)

\[
 z = 0, \; u = 0, \; \frac{8e}{\alpha^2 L^2} + 0 + A_3 + 0 = 0, \; A_3 = -8 \frac{e}{\alpha^2 L^2} 
\]

(18)

\[
 z = \frac{L}{2}, \; u' = 0, \; -\frac{8e}{\alpha^2 L^2} \alpha \sin(\alpha L/2) + A_2 \alpha \cos(\alpha L/2) + 0 = 0, \; A_2 = 8 \frac{e}{\alpha^2 L^2} \tan(\alpha L/2) 
\]

(19)

From Eqs. (15), (16), (17), (18), and (19) the following formula for \( u \) and \( \phi \) are obtained.

\[
 u = \frac{8e}{\alpha^2 L^2} \cos(\alpha z) + \frac{8e}{\alpha^2 L^2} \tan(\alpha L/2) \sin(\alpha z) - \frac{8e}{\alpha^2 L^2} + 4 \frac{e}{L} \left( \frac{z^2}{L} - z \right) 
\]

(20)

\[
 \phi = \frac{E I_y}{M_o} \frac{8e}{L^2} \left( \cos(\alpha z) + \tan(\alpha L/2) \sin(\alpha z) - 1 \right) 
\]

(21)
To get the maximum lateral deflection parameter \( u_m \) for this laterally unrestrained beam, we substitute \( z = L/2 \) in Eq. (20), obtaining

\[
\begin{align*}
\frac{8}{\alpha^2 L^2} \left( \frac{1}{\cos(\alpha L/2)} - \frac{8}{\alpha^2 L^2} - 1 \right)
\end{align*}
\]

Introducing the parameters in equations (13), to Eq. (22) for the maximum lateral deflection \( u_m \) of the centroidal section at midspan of the beam can be written as

\[
\begin{align*}
u_m = u \left( \frac{L}{2} \right) = e \left\{ \frac{8 \pi^2}{(M_o / M_{cr})^2} \left[ \sec \left( \frac{\pi M_o}{2 M_{cr}} \right) - 1 \right] - 1 \right\}
\end{align*}
\]

while the maximum angle of twist \( \phi_m \) is

\[
\begin{align*}
\phi_m = \phi \left( \frac{L}{2} \right) = 2 e \frac{1}{L M_o / M_{cr}} \left[ \sec \left( \frac{\pi M_o}{2 M_{cr}} \right) - 1 \right]
\end{align*}
\]

Later, the above expressions will be used to reveal the relationship between the maximum lateral deflection and the bending moment for various values of eccentricity.

**Laterally restrained and eccentrically loaded beams**

Here we assume that several deck-ties located along the length of a spandrel beam provide a nearly continuous support sufficient to prevent lateral displacements at the mid-height level of the spandrel as long as the connections remain intact. However, the beam is free to rotate around the horizontal line of the deck-ties. In the case of a deep beam subjected to eccentric loading, such twisting deformation may result in large lateral displacements at the top and bottom of the cross section of the beam, which may put the
safety of the structure at risk. Therefore, the maximum lateral deflection in a laterally
restrained beam under eccentric load (Fig. 9.a) is to be analyzed herein.

Unlike unrestrained beams, laterally restrained beams are subjected to horizontal
reaction forces at their lateral supports, yielding a weak axis bending moment \( M_y \) along
their span length. The moment \( M_y \) is unknown at the beginning of the analysis and varies
along the length of the beam. To obtain the governing differential equations of
equilibrium for this case, we first determine the components of the bending moment
vector \( M_y \) in the \( x^*, y^*, \) and \( z^* \) coordinate system, as shown in Fig. 9.b, and then add
them to the differential equations of equilibrium Eqs (6), (7), and (8), previously derived
for the case of laterally unrestrained beams.

Thus differential equations of equilibrium for restrained beams are

\[
- EI_x y'' = M_x - M_z u' + M_y \phi \\
EI_y u'' = -M_z \phi - M_z v' + M_y \\
GJ \phi' = M_z u' + M_z + M_y v'
\]

An approximate solution of this problem can be obtained by neglecting higher order
terms; \( M_z du/dz, M_y \phi, M_z dv/dz, \) and \( M_y dv/dz \). In general, \( M_y \) and \( M_z \) are relatively small
comparing to \( M_z \) (or \( M_x \)) and their products with the slopes (i.e., \( du/dz \) and \( dv/dz \)) or the
angle of twist \( \phi \) are even smaller. While these assumptions are introduced to obtain an
approximate closed form solution (to be later verified by the finite element analyses),
their introduction can be also rationalized on the physical ground. Namely, for laterally
restrained beams, lateral curvature and thus lateral bending moment are small. Also, the
moment \( M_z \) is dominant in comparison with \( M_x \) (which has been assumed earlier).
Further simplification can be made using the equivalent loading approach, which was previously described for the case of unrestrained beams. Under these assumptions, the differential equations become

\[-EI_y y'' = M_o + M_y \phi\]  
\(M_y = M_o \phi\)  
\(GJ \phi' = \frac{eq_o L}{2} \left( 1 - 2 \frac{z}{L} \right)\)  

It needs to be emphasized that \(M_y\) is an unknown function, which defines the variation of the weak axis bending due to the reaction forces at the lateral supports. Furthermore, the lateral deflection \(u\) disappears in these equations since it is fixed by deck-ties close to the centroidal axis of the beam (true only if deck ties are at the centroidal elevation).

Integrating Eq. (30) with respect to \(z\), and considering that \(\phi=0\) at \(z=0\), we find

\[\phi = \frac{eq_o L}{2GJ} \left( z - \frac{z^2}{L} \right)\]  

The angle of twist attains its maximum value \(\phi_m\) at the mid-span of the beam and is equal to

\[\phi_m = \frac{eq_o L^2}{8GJ}\]  

With the notation specified in equations (13), one has

\[\phi_m = \frac{eM_o}{GJ} = \frac{eM_o}{M_{cr}} \frac{\pi}{L} \sqrt{\frac{EI_y}{GJ}}\]  

Assuming \(G=0.4E, J=hb^3/3\) and \(I_y=hb^3/12\) in the equation above, we find the approximate value of the maximum angle of twist
\[
\phi_m = 2.5 e \frac{M_o}{L M_{cr}}
\]  

(34)

**Bending moment-lateral deflection curves**

In the above discussion of laterally unrestrained beams, the maximum lateral deflection \(u_m\) of the centroidal line of the beam was calculated. However, due to twisting (see Fig. 3), the lateral deflection \(u_m\) is smaller than the deflection at the top of the section \(u_{mt}\) and larger than that at the bottom \(u_{mb}\). For serviceability considerations, it is necessary to find the magnitude of the maximum deflection. Considering that deformation of the analyzed beam involves twist, the lateral deflections vary linearly along the depth of the beam. To describe this variation, the top and bottom deflections are computed

\[
u_{mt} = u_m + \phi_m \frac{h}{2} \quad \text{and} \quad u_{mb} = u_m - \phi_m \frac{h}{2}
\]

(35)

where \(h\) is the height of the cross section. The maximum lateral deflection occurs at the top of the mid-span section and its magnitude can be calculated using the following equation, derived by substituting Eqs. (23) and (24) in Eq. (35).

\[
u_{mt} = u_m + \phi_m \frac{h}{2} = e^{\left[ -\sec \left( \frac{\pi M_o}{2 M_{cr}} \right) - 1 \right] \frac{8 / \pi^2}{M_o / M_{cr} + h} - 1} \left[ \frac{M_o / M_{cr}}{L} - 1 \right]
\]

(36)

However, in the case of laterally restrained beams, lateral deflections of the beam at its mid-height level are not permitted. Therefore, due to twisting of the beam, as illustrated
in Fig. 3, lateral deflections are developed at the top and bottom of the section (i.e., $u_{mt}$ and $u_{mb}$). These deflections are equal, but in opposite directions and computed as follows:

$$u_{mt} = \phi_m \frac{h}{2} \quad \text{and} \quad u_{mb} = -\phi_m \frac{h}{2} \quad (37)$$

Using Eq. (34), derived for the angle of twist at the mid-span of the beam, the maximum lateral deflections at the top and bottom of the cross section of a restrained beam are:

$$u_{mb} = -\phi_m \frac{h}{2} = e \left( -1.25 \frac{M_o}{M_{cr}} \frac{h}{L} \right) \quad (38)$$

Using Eqs. (36) and (38) with the height-to-length ratio $h/L=0.11$, relationships between maximum lateral deflections ($u_{mt}$ and $u_{mb}$) and bending moment $M_o$ for laterally unrestrained and restrained beams subjected to eccentric loading are shown in Fig. 10 for various values of eccentricity ratios $e/L$ equal to 0.002, 0.007, and 0.015. Positive deflections $u_{mt}$ occur for laterally unrestrained beams, which means that beams moves toward the double-tees. These curves, on the positive region of the graph, illustrate the high sensitivity of the unrestrained beam to the eccentricity of the load, with lateral deflection, $u_{mt}$, growing in an unbounded manner as the critical value for bending moment $M_o$ is approached. However, this critical moment $M_{cr}$, which corresponds to the horizontal asymptote for the family of curves shown in Fig. 10, is the same for all values of $e/L$. Torsional effect due to the eccentricity, in fact, serves merely to perturb the relationship between $M_o$ and $u_{mt}$ in the same manner as would an initial imperfection (i.e., out-of-straightness). Thus, the torsional effect serves only to increase the lateral
deflection of the beam before it buckles (i.e., when it is still in a theoretically stable configuration).

In Fig. 10, negative deflections $u_{mb}$ are noted for the laterally restrained beams. The bottom of the mid-span section of the beam moves outward. The relationships between lateral deflections $u_{mb}$ and bending moment $M_o$ are shown for different eccentricity ratios. With an increase of the eccentricity, magnitudes of lateral deflections increase for restrained beams, but still remain much smaller than those for unrestrained beams for the same $M_o/M_{cr}$ ratio. Based on Eq. (38), it can also be concluded that lateral deflections are proportional to both the height-to-length ratio $h/L$ and the eccentricity $e$.

In order to compare lateral deflections at the centroid and top of the mid-span section of a laterally unrestrained beam, bending moment-lateral deflection curves for various eccentricity parameters were plotted in Fig. 11, using Equations (23) and (36), respectively. The figure shows that the centroid of the section undergoes slightly smaller lateral deflections than the top of the section due to the twisting of the beam. Even tough the lateral distance between these two points at the section increases with an increase of eccentricity, twisting of the beam contributes slightly to the maximum lateral deflection.

The effect of slenderness ratio $h/L$ on the maximum lateral deflection (at the top) in a laterally unrestrained beam is also investigated. The slenderness ratio $h/L$ of beams is assumed to range from 0.08 to 0.24. Fig. 12 shows bending moment-lateral deflections curves for various eccentricity parameters ($e/L=0.002$, 0.007 and 0.015). Bending moment-lateral deflection curves for beams with small values of eccentricity parameters (i.e., $e/L=0.002$) are not significantly affected by different slenderness ratios. The lateral
response of an unrestrained beam is more sensitive to its slenderness ratio if the eccentricity increases (i.e., $e/L=0.015$).

**Finite Element Analysis**

Approximate closed-form solutions, derived above, for lateral deflections in laterally restrained and unrestrained beams subjected to eccentric loading, involved various modifications of the original problem such as neglecting some terms in the governing equations and using modified loads. These simplifications were necessary to provide approximate solutions for differential equations involved. In order to investigate the accuracy of the approximate solutions, the results were compared with those obtained with finite element analyses. Using commercial finite element (FE) software ABAQUS, three dimensional models of laterally restrained and unrestrained elastic spandrel beams were generated.

Each of restrained and unrestrained beams was analyzed with three different finite element models. The first model uses beam elements in three dimension and is subjected to the equivalent loads (i.e., uniformly distributed torque $m_o=eq_o=eq/C_b$ and end moments $M_o=q_oL^2/8$). The second model is again based on beam elements in three dimensions, but subjected to actual loads (uniformly distributed load acting eccentrically). The third model uses three-dimensional 8-node solid brick elements under uniformly distributed load at the mid-height level of the beam with the eccentricity $e=b/2$. Given the symmetry of the beam, only one half of the beam is modeled. In all models, the beams consist of simple supports at their ends, as well as torsional constraints. Lateral restraints are
provided by fixing the horizontal displacements of the nodes along the centerline of the beam elements in the first and second models and along the line, at half depth, on the loaded side of the third model. The material model for prestressed concrete was assumed to be linearly elastic. Large deformations were included and solved using the Modified Riks Method (Arc-length method) as a solution procedure for FE analyses.

Geometrical and mechanical properties of the beam selected for this numerical study are: $L=534$ in.; $h=60$ in.; $b=8$ in.; $e=4$ in.; $I_x=144,000$ in$^4$; $I_y=2,560$ in$^4$; $J=10,240$ in$^4$; $E=4,800$ ksi; $G=1,920$ ksi. The critical buckling moment $M_{cr}$ is calculated as 91,440 k-in using Eq. (4) for the beam subjected to fictitious loading (end bending moments). However, the critical buckling moment $M_{cr}$ increases by the factor of $C_b$ when the actual loads (uniformly distributed load) act on the beam, and it is equal to 103,330 k-in for $C_b=1.13$ which corresponds to uniformly distributed loads. Results are presented using dimensionless parameters, the moment ratio $M_o/M_{cr}$ where $M_o$ is the maximum bending moment in the beam (i.e., end moments in the first model and $qL^2/8$ in other models) and the deflection parameter $u/L$.

Fig. 13 shows the relationship between the moment ratio $M_o/M_{cr}$ and the deflection parameter $u/L$ for each of the analyzed models of restrained and unrestrained beams. Analytical results obtained using Eqs. (36) and (38) are also shown as solid lines in this figure. Lateral deflections at the top of the mid-span section of the laterally unrestrained beams are in the positive direction (toward the loads), whereas lateral deflections at the bottom of the mid-span section of the laterally restrained beams are in the negative direction. Fig. 13 indicates that numerical and analytical results are in very good
agreement for both restrained and unrestrained beams. In fact the analytical results closely match the numerical results from the model under the equivalent loading, which proves that neglecting higher order terms when finding the analytical solutions does not lead to significant error. For laterally unrestrained beams, Fig. 13 also shows that there is a minor discrepancy between actual and equivalent loading cases. Consequently, it is seen that the equivalent loading is a powerful tool to estimate the lateral deflections in laterally restrained and unrestrained slender beams under uniformly distributed eccentric loads. It should be noted that the \( e/L \) and \( h/L \) ratios for the beam studied here were 0.007 and 0.11, respectively, and similar numerical and analytical responses were observed for other \( e/L \) and \( h/L \) ratios, not shown here for brevity.

**Conclusions**

Slender, precast and prestressed concrete spandrel beams under eccentric loading are susceptible to large lateral deformations and possible serviceability failure before reaching their stability or strength limits. Depending of the strength and reliability of deck-tie connections, spandrel beams were classified as: (a) an unrestrained beam, which is free to deform laterally, and (b) a restrained beam, which was assumed to have a continuous lateral support along the centroidal line of the beam. In this study, approximate analytical solutions for maximum lateral deflections in laterally restrained and unrestrained rectangular beams under eccentric loading were derived exploring a simplified version of second-order elastic analysis.
To obtain a simple closed-form approximate solution for the governing differential equations of the problem, the equivalent loading approach is proposed, in which the eccentric load (actual load) is replaced with end bending moments and uniform torque (equivalent loads), properly calculated. After finding the maximum angle of twist in restrained and unrestrained beams, it has been shown that (a) the maximum lateral deflection occur at the top of the mid-span section of the laterally unrestrained beam, (b) all sections in unrestrained beams under eccentric loads laterally move toward the loading side, (c) laterally restrained beam undergoes only twisting due to the eccentricity, or rotation about its longitudinal direction, (d) as expected, deflections of restrained beams are much smaller than those of unrestrained beams, however, the difference is so big that it seems practical to pay increased attention to the design of deck-ties (e) lateral deflection at the bottom of restrained beam moves outward, which can cause the double-tee beams to lose their supports. Numerical solutions were also provided from three-dimensional finite element analyses and their results were found to be closely comparable with those of analytical solutions.
Notation

\[ b = \] width of beam
\[ C_b = \] bending coefficient
\[ e = \] eccentricity
\[ E = \] elastic modulus
\[ E_c = \] elastic modulus of concrete
\[ E_r = \] reduced elastic modulus of concrete
\[ G = \] elastic shear modulus
\[ h = \] height of beam
\[ I_x = \] moment of inertia about the strong axis
\[ I_y = \] moment of inertia about the weak axis
\[ J = \] torsional constant
\[ L = \] length of beam
\[ m = \] uniformly distributed torsion
\[ M_{cr} = \] critical bending moment
\[ M_i = \] moment in global axes for \( i=x,y,z \); local axes for \( i=x^*,y^*,z^* \)
\[ M_0 = \] bending moment couples about strong-axis at the ends of beam
\[ m_o = \] equivalent distributed torsion
\[ q = \] uniformly distributed load
\[ q_o = \] equivalent distributed load
\[ u = \] lateral displacement
\[ u_m = \] maximum lateral displacement of beam centerline
\[ u_{mb} = \] maximum lateral displacement at the bottom of beam section
\[ u_{mt} = \] maximum lateral displacement at the top of beam section
\[ v = \] vertical displacement
\[ \phi = \] angle of twist
\[ \phi_m = \] maximum angle of twist
References


Klein, G. J. (1986). *Design of Spandrel Beams*, Research Project No. 5, PCI, Chicago, IL.


Fig. 1. Precast and prestressed concrete spandrel beams
Fig. 2. The spandrel-to-double-tee connection
Fig. 3. Deformations of laterally restrained and unrestrained beams (double-tee beams align in the positive $x$-axis)
Fig. 4. (a) Global and local coordinate systems and (b) sign convention for positive internal moments
Fig. 5. Deformations in x-y, y-z and z-x planes
Fig. 6. Components of bending moment $M_x$ along local axes, $x^*, y^*$ and $z^*$.
Fig. 7. (a) Laterally unrestrained beam under eccentric loading and (b) components of moment $M_z$ in the local axes.
Fig. 8. Actual and equivalent loadings for laterally unrestrained beam
Fig. 9. (a) Laterally restrained beam subjected to eccentric loading and (b) components of moment $M_y$ in the local axes
Fig. 10. Bending moment-maximum lateral deflection curves for various values of eccentricity ($h/L=0.11$)
Fig. 11. Lateral deflections at the top and centroid of the laterally unrestrained beams

\((h/L=0.11)\)
Fig. 12. The response of unrestrained beams for different slenderness ratios ($h/L=0.08, 0.24$)
Fig. 13. Numerical vs. analytical results for maximum lateral deflections (\(e/L=0.007\), 
\(h/L=0.11\))